Mid-Term Exam
Due Nov 2, 2005

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Important Points to Note

- This exam is due Nov 2, 2005 at 12:30pm sharp in Prof. Chakrabarti's mailbox. Late penalties will be assessed exactly as for the homeworks.
- Please write or type your solutions neatly and staple together your sheets of paper. We are not responsible for sheets lost due to lack of stapling.
- You must work on the exam *alone*; you may not collaborate with anyone.
- You may consult your textbook (Sipser), your notes, and anything posted on the CS 39 website. Consulting anything else (e.g. last year's notes, your friend's notes, other websites) is a violation of the Honor Code.
- If a problem does not ask for a proof, you are not required to provide one.
- If a problem *does* ask for a proof, you *must provide a formal mathematical proof*; intuition is all very well, but a proof which is basically a lengthy essay in plain English with no accompanying mathematics will get very little credit. You may use, without proof, any results proven in class, in a homework, or in the textbook.
- Please read each question carefully. Unfortunately, if you misread and answer a different question than the one asked, you will not get credit.
- You may speak to others about the exam only in *complete generality* (e.g., "The exam is hard", "I'm almost finished with the exam", "I'll be working on the exam tonight"). You may not speak about the exam in any detail whatsoever (e.g., "Problem 3 is hard", "Problem 5 is easy", "That pumping lemma problem is tough").
- Since this is an exam, I cannot help you with the particular problems on this exam. However, as you attempt to solve these problems, if you discover that your understanding is not complete on some topics, please see me. I am willing to help you with those concepts to any degree. But please don't ask me to check if you are on the right track with a problem.
- Good luck!

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1. Write a regular expression for the language generated by the following grammar:

$$\begin{aligned} S &\longrightarrow AT \\ T &\longrightarrow ABT \mid TBA \mid AA \\ A &\longrightarrow 0 \\ B &\longrightarrow 1 \end{aligned}$$

A single line answer will do; you don't have to justify or show any steps. Your regular expression should be as simple as possible.

[5 points]

2. Draw a DFA for the language

$$\{x \in \{0,1\}^* : x \text{ contains an equal number of occurrences of the substrings } 01 \text{ and } 10\}.$$

For example, 101 and 0000 are in the language, but 1010 is not.

[5 points]

3. Recall that $x^{\mathcal{R}}$ denotes the reverse of the string x. For a language L, let $L^{\mathcal{R}} = \{x^{\mathcal{R}} : x \in L\}$. Prove that if L is regular, so is $L^{\mathcal{R}}$.

[10 points]

4. Let L_1, L_2 be two languages over the same alphabet Σ. Prove that $(L_1^* L_2^*)^* = (L_1 \cup L_2)^*$. Remember that to prove $A = B$ for sets A and B you must separately prove the two statements $A \subseteq B$ and $B \subseteq A$.

[8 points]

5. Prove that there exist languages $A, B, C \subseteq \{0,1\}^*$ that satisfy all of the following properties:

(a) $A = B \cap C$.

(b) B and C are both non-regular.

(c) A is infinite and regular.

To get any credit, you *must* prove *all* three properties for whatever A, B, C you have decided to use.

[12 points]

6. A permutation of a string x is any string that can be obtained by rearranging the characters of x. Thus, for example, the string abc has exactly six permutations:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba\,.$$

Clearly, if y is a permutation of x, then $|y| = |x|$. For a language L over alphabet Σ, define

$$\begin{aligned} \textsc{Permute}(L) &= \{x \in \Sigma^* : x \text{ is a permutation of some string in } L\}\,, \\ \textsc{Select}(L) &= \{x \in \Sigma^* : \text{every permutation of } x \text{ is in } L\}\,. \end{aligned}$$

Classify each of the following statements as TRUE or FALSE, and give proofs justifying your classifications.

6.1. If $L_1 = 1^*0$, then $\textsc{Permute}(L_1)$ is regular.

[5 points]

6.2. If $L_1 = 1^*0$, then $\textsc{Select}(L_1)$ is regular.

[5 points]

6.3. Regular languages are closed under the operation PERMUTE.

[10 points]

6.4. Regular languages are closed under the operation SELECT.

[10 points]

7. Draw a PDA for the language $\{0^i 1^j : i < j < 2i\}$. For clarity, keep your stack alphabet disjoint from $\{0, 1\}$. Provide a brief justification (no need for a formal proof) that your PDA works correctly.

[15 points]

8. Design a context-free grammar for the *complement* of the language $\{a^n b^n : n \geq 0\}$ over the alphabet $\{a, b\}$. Give brief explanations for the "meanings" of your variables (i.e. explain what strings are generated by each of your variables).

[15 points]

Here endeth the exam.