1. Write a regular expression for the language generated by the following grammar:

\[
\begin{align*}
S & \rightarrow AT \\
T & \rightarrow ABT \mid TBA \mid AA \\
A & \rightarrow 0 \\
B & \rightarrow 1
\end{align*}
\]

A single line answer will do; you don't have to justify or show any steps. Your regular expression should be as simple as possible.

[5 points]

2. Draw a DFA for the language

\[
\{ x \in \{0,1\}^* : x \text{ contains an equal number of occurrences of the substrings 01 and 10} \}.
\]

For example, 101 and 0000 are in the language, but 1010 is not.

[5 points]
3. Recall that $x^R$ denotes the reverse of the string $x$. For a language $L$, let $L^R = \{x^R : x \in L\}$. Give a complete formal proof that if $L$ is regular, so is $L^R$.  

[10 points]

4. Consider two languages $A, B \subseteq \Sigma^*$. Prove that $(A^*B^*)^* = (A \cup B)^*$. Remember that to prove $X = Y$ for sets $X$ and $Y$ you must separately prove $X \subseteq Y$ and $Y \subseteq X$.  

[8 points]

5. Prove that there exist languages $A, B, C \subseteq \{0,1\}^*$ that satisfy all of the following properties:

(a) $A = B \cap C$.
(b) $B$ and $C$ are both non-regular.
(c) $A$ is infinite and regular.

To get any credit, you must prove all three properties for whatever $A, B, C$ you have decided to use.  

[12 points]

6. A permutation of a string $x$ is any string that can be obtained by rearranging the characters of $x$. Thus, for example, the string $abc$ has exactly six permutations:

$abc, acb, bac, bca, cab, cba$.

Clearly, if $y$ is a permutation of $x$, then $|y| = |x|$. For a language $L$ over alphabet $\Sigma$, define

$$\text{PERMUTE}(L) = \{ x \in \Sigma^* : x \text{ is a permutation of some string in } L \},$$
$$\text{SELECT}(L) = \{ x \in \Sigma^* : \text{every permutation of } x \text{ is in } L \}.$$ 

Classify each of the following statements as TRUE or FALSE, and give proofs justifying your classifications.

6.1. If $L_1 = 1^*0$, then $\text{PERMUTE}(L_1)$ is regular.  

[5 points]

6.2. If $L_2 = 0^*1^*$, then $\text{SELECT}(L_2)$ is regular.  

[5 points]

6.3. Regular languages are closed under the operation $\text{PERMUTE}$.  

[10 points]

6.4. Regular languages are closed under the operation $\text{SELECT}$.  

[10 points]
7. Draw a PDA for the language \( \{0^i1^j : i < j < 2i \} \). For clarity, keep your stack alphabet disjoint from \( \{0, 1\} \). Provide a brief justification (no need for a formal proof) that your PDA works correctly.

[15 points]

8. Design a context-free grammar for the complement of the language \( \{a^n b^n : n \geq 0 \} \) over the alphabet \( \{a, b\} \). Give brief explanations for the “meanings” of your variables (i.e. explain what strings are generated by each of your variables).

[15 points]

Here endeth the exam.