Homework 3 Solutions
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1. **(DFA → regular expression)**

   Throughout this solution, we shall use the shorthand R^+ for RR^*, where R is an arbitrary regular expression.

   1.1. $R_{11}^0 = \varepsilon \cup a$
   $R_{12}^0 = b$
   $R_{21}^0 = b$
   $R_{22}^0 = \varepsilon \cup a$

   $R_{11}^1 = R_{11}^0 \cup R_{11}^0(R_{11}^0)^* R_{11}^0 = (\varepsilon \cup a)^+ = a^*$
   $R_{12}^1 = R_{12}^0 \cup R_{11}^0(R_{11}^0)^* R_{12}^0 = b \cup (\varepsilon \cup a)^+ b = a^* b$
   $R_{21}^1 = R_{21}^0 \cup R_{21}^0(R_{11}^0)^* R_{11}^0 = b \cup b(\varepsilon \cup a)^+ = ba^*$
   $R_{22}^1 = R_{22}^0 \cup R_{21}^0(R_{11}^0)^* R_{12}^0 = (\varepsilon \cup a) \cup b(\varepsilon \cup a)^* b = \varepsilon \cup a \cup ba^* b$

   $R_{12}^2 = R_{12}^1 \cup R_{12}^1(R_{22}^1)^* R_{22}^1 = a^* b \cup a^* b(\varepsilon \cup a \cup ba^* b)^+ = a^* b(a \cup ba^* b)^*$
   $\Rightarrow L = R_{12}^2 = a^* b(a \cup ba^* b)^*$

   1.2. $R_{11}^0 = \varepsilon$
   $R_{12}^0 = a \cup b$
   $R_{13}^0 = \phi$
   $R_{21}^0 = \phi$
   $R_{22}^0 = \varepsilon \cup a$
   $R_{23}^0 = b$
   $R_{31}^0 = a$
   $R_{32}^0 = b$
   $R_{33}^0 = \varepsilon$

   $R_{11}^1 = \varepsilon$
   $R_{12}^1 = a \cup b$
   $R_{13}^1 = \phi$
   $R_{21}^1 = \phi$
   $R_{22}^1 = \varepsilon \cup a$
   $R_{23}^1 = b$
   $R_{31}^1 = a$
   $R_{32}^1 = b \cup a(a \cup b) = b \cup aa \cup ab$
   $R_{33}^1 = \varepsilon$

   $R_{11}^2 = \varepsilon$
   $R_{12}^2 = a \cup b \cup (a \cup b)a^* = a^+ \cup ba^*$
   $R_{13}^2 = (a \cup b)a^* b = a^+ b \cup ba^* b$
   $R_{21}^2 = \phi$
   $R_{22}^2 = a^*$
   $R_{23}^2 = b \cup (\varepsilon \cup a)a^* b = a^* b$
   $R_{31}^2 = a$
   $R_{32}^2 = (b \cup aa \cup ab) \cup (b \cup aa \cup ab)a^*(\varepsilon \cup a) = ba^* \cup aa^+ \cup aba^*$
   $R_{33}^2 = \varepsilon \cup (b \cup aa \cup ab)a^* b = \varepsilon \cup ba^* b \cup aa^+ b \cup aba^* b$

   $R_{11}^3 = \varepsilon \cup (a^+ b \cup ba^* b)(ba^* b \cup aa^+ b \cup aba^* b)^* a$
   $R_{13}^3 = (a^+ b \cup ba^* b)(ba^* b \cup aa^+ b \cup aba^* b)^*$

$\Rightarrow L = R^3_{11} \cup R^3_{13}$

2. **(True or false)**

2.1. False. For a counterexample, let L be any non-regular language such as $\{0^n1^n : n \geq 0\}$. Since L is non-regular, $\overline{L}$ (the complement of L) is also nonregular. Yet, $L \cup \overline{L} = \Sigma^*$ is regular.

2.2. False. For a counterexample, let L be any non-regular language such as $\{0^n1^n : n \geq 0\}$. Since L is non-regular, $\overline{L}$ (the complement of L) is also nonregular. Yet, $L \cap \overline{L} = \Phi$ is regular.

2.3. True. We proved in class that if a language is regular, so is its complement. This is equivalent to the statement that if the complement of a language is regular, so is that language itself.

2.4. False. Any set can be written as a (possibly infinite) union of singleton sets containing its elements. In particular, any language L can be written as a union of finite, therefore regular, languages: $L = \bigcup_{x \in L} \{x\}$. More concretely, take our favorite nonregular language. We have $\{0^n1^n : n \geq 0\} = \bigcup_{n=0}^{\infty} \{0^n1^n\}$.

2.5. False. If this were true, then by De Morgan's Law the previous would also have to be true. For a concrete counterexample, let $A_n = \{0^n1^n\}$ for every $n \geq 0$. Then for every n, $\overline{A_n}$ is regular. Assume, to get a contradiction, that the statement is true. Then $\bigcap_{n=0}^{\infty} \overline{A_n}$ is regular, so that $\overline{\bigcap_{n=0}^{\infty} \overline{A_n}}$ is regular. But by De Morgan's Law, the latter is just $\bigcup_{n=0}^{\infty} A_n$, which we know to be nonregular, giving us our contradiction.

3. **(L regular $\Rightarrow$ MAX(L) regular)**

If $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA for L, then the intuition to construct a DFA M' for MAX(L) is as follows. If q_f is a final state of M and there is a non-empty string that drives M from q_f to a final state (possibly q_f itself), then q_f should not be a final state in M'. This ensures that M' does not accept a string in L if there is a way of extending it to be another string in L.

Formally, let $M' = (Q, \Sigma, \delta, q_0, F')$, where $F' = \{q : q \in F \text{ and } \forall x \in \Sigma^+, \hat{\delta}(q, x) \notin F\}$.

4. **(L regular $\Rightarrow$ CYCLE(L) regular)**

We observe that a string w is in CYCLE(L) if and only if there is a way to split w into two parts: x_1 and x_2, such that there is a state q of L's DFA M satisfying

1. $\hat{\delta}(q, x_1) \in F$ and
2. $\hat{\delta}(q_0, x_2) = q$.

That is to say, a marble starting off in state q ends up in a final state of M upon consuming x_1, and a marble starting off in the initial state of M ends up in q upon consuming x_2. This suggests that the marble should keep track of three things: (1) the state of M where it started, (2) the state of M at which it currently is, and (3) whether it is consuming x_1 or x_2. Accordingly, each state of the new NFA M' will be a 3-vector (p, q, i), where p and q are states of M, and $i \in \{1, 2\}$.

Formally, if $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA for L, define a new NFA $M' = (Q', \Sigma, \delta', q'_0, F')$, where

$$
\begin{aligned}
Q' &= Q \times Q \times \{1,2\} \cup \{q'_0\} \text{ where } q'_0 \notin Q \times Q \times \{1,2\} \\
F' &= \{(q, q, 2) : q \in Q\} \\
\delta'(q'_0, \varepsilon) &= \{(q, q, 1) : q \in Q\} \\
\delta'((p, q, 1), \varepsilon) &= \{(p, q_0, 2)\} \text{ for every } q \in F \\
\delta'((p, q, i), a) &= \{(p, \delta(q, a), i)\} \text{ if } a \in \Sigma
\end{aligned}
$$

By the discussion above, M' recognizes CYCLE(L).

5. **(Regular or not?)**

5.1. $L = \{0^m 1^n 0^{m+n} : m, n \geq 0\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 1^p 0^p$, where p is the pumping length. Clearly, $s \in L$ (for $m = 0$ and $n = p$) and $|s| \geq p$, so let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| \leq p$, y must lie entirely within the sequence of 1's. Hence, $xz = 1^{p-|y|}0^p$ should belong to L by the Lemma, but it does not since $p - |y| \neq p$, giving us our contradiction.

5.2. $L = \{0^m 1^n : m \text{ divides } n\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 0^p 1^p$, where p is the pumping length. Again, let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| \leq p$, y must lie entirely within the sequence of 0's. Hence, $xy^2z = 0^{p+|y|}1^p$ should belong to L by the Lemma, but does not since $p + |y| > p$ so it certainly does not divide p, giving us our contradiction.

5.3. $L = \{xwx^R : x, w \in \{0,1\}^* \text{ and } |x|, |w| > 0\}$.
Regular. Careful observation will reveal that a string is in L if and only if it starts and ends with the same symbol and is of length at least three. L is therefore captured by the regular expression $0(0 \cup 1)^+0 \cup 1(0 \cup 1)^+1$.

5.4. $L = \{0^{2^n} : n \geq 0\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 0^{2^p}$, where p is the pumping length. Let $s = xyz$ as specified by the Pumping Lemma. Then by the Lemma, $xy^2z \in L$. However, clearly $|xy^2z| > |xyz| = 2^p$, yet $|xy^2z| < 2^{p+1}$ since $|y| \leq |xy| \leq p < 2^p$, so $xy^2z \notin L$, giving us our contradiction.

5.5. $L = \{w \in \Sigma_2^* : \text{the bottom row of } w \text{ is the reverse of the top row of } w\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = \begin{bmatrix}0\\0\end{bmatrix}^p \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}^p$, where p is the pumping length. Let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| < p$, y lies entirely in the first sequence of $\begin{bmatrix}0\\0\end{bmatrix}$'s. Hence, $xz = \begin{bmatrix}0\\0\end{bmatrix}^{p-|y|} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}^p$ does not belong to L, contradicting the Lemma.

5.6. $L = \{0^m 1^n : m, n \geq 0 \text{ and } m \neq n\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let R denote the language captured by the regular expression 0^*1^*. Then $L \cup \overline{R}$, and therefore $\overline{L \cup \overline{R}}$, must be regular since the set of all regular languages is closed under union and complementation. But the latter expression is precisely the language $\{0^n 1^n : n \geq 0\}$, which we know to be nonregular, giving us our contradiction.