Please think carefully about how you are going to organise your answers before you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Do both parts of Exercise 1.21 from the textbook. Use the \( R_{ij}^k \) method as described in the lecture notes for the Oct 12 lecture; do not use the textbook’s “GNFA method.” Try to simplify the intermediate regular expressions, so as to make your own life simple! [6+7 points]

2. Are the following statements always true? If true, give a brief justification and if false, give a concrete counterexample. Below, \( A \) and \( B \) denote languages over some alphabet \( \Sigma \).

   2.1. If \( A \cup B \) is regular, then at least one of \( A \) and \( B \) is regular. [5 points]

   2.2. If \( A \cap B \) is regular, then at least one of \( A \) and \( B \) is regular. [5 points]

   2.3. If \( \overline{A} \) (defined as \( \Sigma^* - A \)) is regular, then \( A \) is regular. [5 points]

   2.4. A union of arbitrarily many regular languages is regular, even if it is an infinite union. [5 points]

   2.5. An intersection of arbitrarily many regular languages is regular, even if it is an infinite intersection. [5 points]

3. For a language \( L \) over alphabet \( \Sigma \), define \( \text{MAX}(L) = \{ x \in \Sigma^* : x \in L \text{ and } x \text{ is not a proper prefix of any string in } L \} \). Prove that if \( L \) is regular, then so is \( \text{MAX}(L) \). [8 points]

4. For a language \( L \) over alphabet \( \Sigma \), define \( \text{CYCLE}(L) = \{ x_1x_2 : \exists x_1, x_2 \in \Sigma^* \text{ such that } x_2x_1 \in L \} \). Prove that if \( L \) is regular, then so is \( \text{CYCLE}(L) \). [12 points]
5. For each of the following languages, say whether or not the language is regular and prove your answer. To prove that a language is regular, specify a finite automaton or a regular expression for that language. To prove that a language is not regular, use the pumping lemma or closure properties of regular languages.

Proofs must be precisely written. Make sure you fully understand the definitions of the sets before answering.

5.1. \( \{0^m1^n0^{m+n} : m, n \geq 0\} \). [7 points]

5.2. \( \{0^m1^n : m \text{ divides } n\} \). [7 points]

5.3. \( \{xwx^R : x, w \in \{0, 1\}^*, |x| > 0 \text{ and } |w| > 0\} \). [7 points]

5.4. \( \{0^{2^n} : n \geq 0\} \). [7 points]

5.5. Problem 1.35 from the textbook. [7 points]

5.6. \( \{0^m1^n : m, n \geq 0 \text{ and } m \neq n\} \). [7 points]

**Challenge Problems**

**CP3:** Let \( L \) be any subset of \( 0^* \). Prove that \( L^* \) is regular.

This is a delightful problem and will teach you something nice about regular languages if you solve it.