In class, we gave constructions of appropriate finite automata to prove that regular languages are closed under the operations of union and concatenation. These constructions are similar to those in your textbook. We gave informal proofs for why the constructions work, but not formal proofs. Here is how one should formally prove the correctness of the constructions.

**Theorem 1:** If $L_1$ and $L_2$ are regular, so is $L_1 \cup L_2$.

**Proof:** For $1 \leq i \leq 2$, let $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ be a DFA that recognizes $L_i$, and assume that $Q_1 \cap Q_2 = \emptyset$. Define a new state $q_0 \notin Q_1 \cup Q_2$ and an NFA $M = (Q, \Sigma, \delta, q_0, F)$, where

$$Q = Q_1 \cup Q_2 \cup \{q_0\},$$

$$F = F_1 \cup F_2,$$

and $\delta$ is given by

$$\delta(q, a) = \begin{cases} 
\{q_1, q_2\}, & \text{for } q = q_0, a = \varepsilon, \\
\emptyset, & \text{for } q \in Q_1 \cup Q_2, a = \varepsilon, \\
\emptyset, & \text{for } q = q_0, a \in \Sigma, \\
\{\delta_1(q, a)\}, & \text{for } q \in Q_1, a \in \Sigma, \\
\{\delta_2(q, a)\}, & \text{for } q \in Q_2, a \in \Sigma.
\end{cases}$$

We claim that $L(M) = L_1 \cup L_2$. To prove this equality between sets, we must show two containments, as follows.

(1) $L(M) \subseteq L_1 \cup L_2$: Suppose $x \in L(M)$, i.e., $M$ accepts $x$. By definition, this means that we can write $x = a_1 a_2 \cdots a_n$, where each $a_i \in \Sigma$, and there exists a sequence $(r_0, r_1, \ldots, r_n)$ of states in $Q$ such that the following three conditions hold.

- (1.1) $r_0 = q_0$,
- (1.2) $r_{i+1} \in \delta(r_i, a_{i+1})$, $\forall i$ with $0 \leq i \leq n - 1$,
- (1.3) $r_n \in F$.

By conditions (1.2) and (1.1), we have $r_1 \in \delta(r_0, a_1) = \delta(q_0, a_1)$. Therefore, by our construction of $M$, we must have $a_1 = \varepsilon$ and $r_1 \in \{q_1, q_2\}$.

Consider the case when $r_1 = q_1$. Then $r_1 \in Q_1$. Repeatedly applying condition (1.2), we see that for all $i$ with $0 \leq i \leq n - 1$, we have $a_{i+1} \in \Sigma$, $r_{i+1} \in Q_1$ and $r_{i+1} = \delta(r_i, a_{i+1})$. Finally, by condition (1.3), we have $r_n \in F = F_1 \cup F_2$, but we also have $r_n \in Q_1$; therefore, $r_n \in F_1$. We conclude that $x = a_2 a_3 \cdots a_n$ and that the sequence $(r_1, r_2, \ldots, r_n)$ satisfies

- (2.1) $r_1 = q_1$,
- (2.2) $r_{i+1} = \delta(r_i, a_{i+1})$, $\forall i$ with $1 \leq i \leq n - 1$,
- (2.3) $r_n \in F_1$.

By definition, this means that $M_1$ accepts $x$, so $x \in L_1$. Therefore $x \in L_1 \cup L_2$.

Consider the case when $r_1 = q_2$. By a similar argument, we conclude that $x \in L_2$. Therefore $x \in L_1 \cup L_2$.

Thus, in all cases, $x \in L(M)$ implies $x \in L_1 \cup L_2$. We conclude that $L(M) \subseteq L_1 \cup L_2$.

(2) $L_1 \cup L_2 \subseteq L(M)$: Think about how you would prove this. I leave it as an exercise.
**Theorem 2:** If \( L_1 \) and \( L_2 \) are regular, so is \( L_1L_2 \).

**Proof:** For \( 1 \leq i \leq 2 \), let \( M_i = (Q_i, \Sigma, \delta_i, q_i, F_i) \) be a DFA that recognizes \( L_i \), and assume that \( Q_1 \cap Q_2 = \emptyset \). Define an NFA \( M = (Q, \Sigma, \delta, q_1, F_2) \), where

\[
Q = Q_1 \cup Q_2,
\]

and \( \delta \) is given by

\[
\delta(q, a) = \begin{cases} 
\{\delta_1(q, a)\}, & \text{for } q \in Q_1, a \in \Sigma, \\
\{\delta_2(q, a)\}, & \text{for } q \in Q_2, a \in \Sigma, \\
\{q\}, & \text{for } q \in F_1, a = \varepsilon, \quad \text{(*)} \\
\emptyset, & \text{for } q \in (Q_1 - F_1) \cup Q_2, a = \varepsilon.
\end{cases}
\]

We claim that \( \mathcal{L}(M) = L_1L_2 \). To prove this equality between sets, we must show two containments, as follows.

1. \( \mathcal{L}(M) \subseteq L_1L_2 \): Suppose \( x \in \mathcal{L}(M) \), i.e., \( M \) accepts \( x \). By definition, this means that we can write \( x = a_1a_2 \cdots a_n \), where each \( a_i \in \Sigma \), and there exists a sequence \( \langle r_0, r_1, \ldots, r_n \rangle \) of states in \( Q \) such that the following three conditions hold.

   - (3.1) \( r_0 = q_1 \),
   - (3.2) \( r_{i+1} \in \delta(r_i, a_{i+1}), \forall i \text{ with } 0 \leq i \leq n-1 \),
   - (3.3) \( r_n \in F_2 \).

Observe that \( r_0 \in Q_1 \) and \( r_n \in Q_2 \). Therefore, there must exist an integer \( j \), with \( 0 \leq j \leq n-1 \), such that \( r_0, r_1, \ldots, r_j \in Q_1 \) and \( r_{j+1} \in Q_2 \). By construction, the only transitions from a state in \( Q_1 \) to a state in \( Q_2 \) are those given by (\( \ast \)). It follows that \( r_j \in F_1, r_{j+1} = q_2 \) and \( a_{j+1} = \varepsilon \). Further, since there are no \( \varepsilon \)-transitions other than those given by (\( \ast \)), it follows that \( a_i \in \Sigma \) for all \( i \neq j \). Now, by construction of \( M \), it follows that \( r_{j+1}, r_{j+2}, \ldots, r_n \in Q_2 \).

We conclude that \( x = a_1a_2 \cdots a_j\varepsilon a_{j+2} \cdots a_n \) and that the sequences \( \langle r_0, r_1, \ldots, r_j \rangle \) and \( \langle r_{j+1}, r_{j+2}, \ldots, r_n \rangle \) satisfy the following sets of conditions.

   - (4.1) \( r_0 = q_1 \),
   - (4.2) \( r_{i+1} = \delta_1(r_i, a_{i+1}), \forall i \text{ with } 0 \leq i \leq j-1 \),
   - (4.3) \( r_j \in F_1 \).

By definition, conditions (4.1)–(4.3) mean that \( M_1 \) accepts \( a_1a_2 \cdots a_j \) and conditions (5.1)–(5.3) mean that \( M_2 \) accepts \( a_{j+2} \cdots a_n \). Therefore, the former string belongs to \( L_1 \) and the latter string belongs to \( L_2 \). Since the concatenation of these two strings is \( x \), we get \( x \in L_1L_2 \). We conclude that \( \mathcal{L}(M) \subseteq L_1L_2 \).

2. \( L_1L_2 \subseteq \mathcal{L}(M) \): As before, I leave this half of the proof as an exercise. Think about it.

\[
* \quad * \quad *
\]

**NOTE:** An alternative way to define \( \delta \) in the proof of Theorem 1 is as follows.

\[
\delta(q_0, \varepsilon) = \{q_1, q_2\}, \\
\delta(q, \varepsilon) = \emptyset, \quad \text{for } q \in Q_1 \cup Q_2, \\
\delta(q_0, a) = \emptyset, \quad \text{for } a \in \Sigma, \\
\delta(q, a) = \{\delta_1(q, a)\}, \quad \text{for } q \in Q_1, a \in \Sigma, \\
\delta(q, a) = \{\delta_2(q, a)\}, \quad \text{for } q \in Q_2, a \in \Sigma.
\]