Please think carefully about how you are going to organise your answers before you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Let $L$ be the language given by the regular expression $(ab \cup aab \cup aba)^*$.

   1.1. Design an NFA for $L$ that has no $\varepsilon$-transitions and has only 4 states. [6 points]

   1.2. Convert the above NFA into a DFA for $L$ by mechanically using the subset construction we studied in class. [10 points]

   1.3. Remove all states that are unreachable from the start state of the resulting DFA, to get a 7-state DFA for $L$. [3 points]

   1.4. If you carefully observe this DFA, you will notice two states that can be replaced by a single state. Do this and draw the resulting DFA. Your final DFA should have exactly 6 states. [7 points]

2. Construct NFAs equivalent to following regular expressions (your NFAs may have $\varepsilon$-transitions):

   2.1. $10 \cup (0 \cup 11)0^+1$ [7 points]

   2.2. $((0 \cup 1)(0 \cup 1))^* \cup ((0 \cup 1)(0 \cup 1)(0 \cup 1))^*$ [7 points]

3. Give regular expressions for the following languages.

   3.1. $\{w \in \{0,1\}^* : w \text{ has three consecutive 0's or three consecutive 1's or both}\}$. [7 points]

   3.2. $\{w \in \{0,1\}^* : w \text{ has three consecutive 0's and three consecutive 1's}\}$. [7 points]

   3.3. The set of strings in $\{0,1\}^*$ with an equal number of 0's and 1's such that no prefix has two more 0's than 1's nor two more 1's than 0's. [10 points]

   3.4. Let us define a valid floating point number as $u.v$, where $u$ and $v$ are (finite) strings of decimal digits (0..9) satisfying the following constraints: (the symbol “.” between $u$ and $v$ is the decimal point.)

   i. Neither $u$ nor $v$ may be $\varepsilon$.
   ii. $u$ can be just 0. If $u$ is not 0, $u$ has no leading 0's.
   iii. $v$ can be just 0. If $v$ is not 0, $v$ has no trailing 0's.

   (Thus, for example, 0.0, 231.0 and 5.608 are valid, but 0.00, 05.68, .65, 12. and 4.5100 are not valid.)

   Give a regular expression for the set of valid floating point numbers described above. You might want to introduce some notation first to keep your expression small and readable. [10 points]
4. Let \( L \) be a nonempty language and \( M \) an NFA that recognizes \( L \). Prove that \( M \) can be converted into an NFA \( M' \) which recognizes the same language \( L \) and has exactly one accept state. Your proof must describe \( M' \) both informally, using plain English, and formally, using mathematical notation. [10 points]

5. For a language \( L \) over alphabet \( \Sigma \), define \( \text{HALF}(L) = \{ x \in \Sigma^* : \exists y \in \Sigma^* (|x| = |y| \text{ and } xy \in L) \} \). Prove that if \( L \) is regular, then so is \( \text{HALF}(L) \). Your proof must be formal; proofs not written in a formal mathematical style get very little credit even if they express the right intuition. [16 points]

Hint: Approach 1: Build an NFA. Nondeterministically guess which state the DFA for \( L \) will end up in after reading \( x \) and nondeterministically guess a \( y \) to append to \( x \). Approach 2: Build a DFA. Work forwards and backwards simultaneously and try to meet in the middle.

Challenge Problems

Remember that challenge problems carry no regular credit, but are intended to provide a higher level of challenge for those who want to think further about the theory of computing.

CP1: For the language \( L \) from Problem 1, prove that it is impossible to design a DFA with 5 or fewer states.

CP2: For a language \( L \) over alphabet \( \Sigma \), define \( \text{LOG}(L) = \{ x \in \Sigma^* : \exists y \in \Sigma^* (|y| = 2|x| \text{ and } xy \in L) \} \). Prove that if \( L \) is regular, then so is \( \text{LOG}(L) \).