Solutions: Homework 3
Prepared by Chien-Chung Huang

1.16(a) Answer:
\[ R_{11}^0 = \varepsilon \cup a, \quad R_{22}^0 = \varepsilon \cup a, \quad R_{12}^0 = b, \quad R_{11}^0 = b. \]
\[ R_{12}^1 = R_{12}^0 \cup R_{11}^0 (R_{11}^0)^* R_{12}^0 = a^*b \]
\[ R_{22}^1 = R_{22}^0 \cup R_{21}^0 (R_{11}^0)^* R_{12}^0 = (\varepsilon \cup a) \cup ba^*b \]
\[ L = R_{12}^2 = R_{12}^1 \cup R_{12}^1 (R_{22}^1)^* R_{12}^1 = a^*b((a \cup \varepsilon)ba^*b)^* \]

1.16(b) \[ R_{11}^0 = \varepsilon, \quad R_{22}^0 = a \cup \varepsilon, \quad R_{33}^0 = \varepsilon, \quad R_{12}^0 = (a \cup b), \quad R_{23}^0 = b, \quad R_{13}^0 = \phi, \quad R_{21}^0 = \phi, \quad R_{32}^0 = b, \]
\[ R_{31}^0 = a. \]
\[ R_{11}^1 = R_{11}^0 \cup R_{11}^0 (R_{11}^0)^* R_{11}^0 = \varepsilon, \]
\[ R_{22}^1 = R_{22}^0 \cup R_{21}^0 (R_{11}^0)^* R_{12}^0 = a \cup \varepsilon, \]
\[ R_{33}^1 = R_{33}^0 \cup R_{31}^0 (R_{11}^0)^* R_{13}^0 = \varepsilon, \]
\[ R_{12}^1 = R_{12}^0 \cup R_{11}^0 (R_{11}^0)^* R_{12}^0 = (a \cup b), \]
\[ R_{23}^1 = R_{23}^0 \cup R_{21}^0 (R_{11}^0)^* R_{13}^0 = b, \]
\[ R_{13}^1 = R_{13}^0 \cup R_{11}^0 (R_{11}^0)^* R_{13}^0 = \phi, \]
\[ R_{21}^1 = R_{21}^0 \cup R_{21}^0 (R_{11}^0)^* R_{11}^0 = \phi, \]
\[ R_{32}^1 = R_{32}^0 \cup R_{31}^0 (R_{11}^0)^* R_{12}^0 = b \cup a(a \cup b), \]
\[ R_{31}^1 = R_{31}^0 \cup R_{31}^0 (R_{11}^0)^* R_{11}^0 = a. \]
\[ R_{11}^2 = R_{11}^1 \cup R_{12}^1 (R_{22}^1)^* R_{21}^1 = \varepsilon, \]
\[ R_{13}^2 = R_{13}^1 \cup R_{12}^1 (R_{22}^1)^* R_{23}^1 = (a \cup b)a^*b, \]
\[ R_{33}^2 = R_{33}^1 \cup R_{32}^1 (R_{22}^1)^* R_{23}^1 = \varepsilon \cup (b \cup a(a \cup b))a^*b, \]
\[ R_{12}^2 = R_{12}^1 \cup R_{12}^1 (R_{22}^1)^* R_{22}^1 = (a \cup b)a^*, \]
\[ R_{21}^2 = R_{21}^1 \cup R_{22}^1 (R_{22}^1)^* R_{21}^1 = \phi, \]
\[ R_{31}^2 = R_{31}^1 \cup R_{32}^1 (R_{22}^1)^* R_{21}^1 = a, \]
\[ R_{22}^2 = R_{22}^1 \cup R_{22}^1 (R_{22}^1)^* R_{21}^1 = a^*, \]
\[ R_{32}^2 = R_{32}^1 \cup R_{32}^1 (R_{22}^1)^* R_{22}^1 = (b \cup a(a \cup b))a^*, \]
\[ R_{11}^3 = \varepsilon \cup (a \cup b)a^*b((b \cup a(a \cup b))a^*b)^*a. \]
\[ R_{13}^3 = (a \cup b)a^*b((b \cup a(a \cup b))a^*b)^*. \]
\[ L = R_{11}^3 \cup R_{13}^3. \]
2.1 Answer: False. For a counterexample, let $L$ be any non-regular language such as $\{0^n1^n \mid n > 0\}$. Since $L$ is non-regular, $\overline{L}$ (the complement of $L$) is also nonregular. Yet, $L \cup \overline{L} = \Sigma^*$ is regular.

2.2 Answer: False. For a counterexample, let $L$ be any non-regular language such as $\{0^n1^n \mid n > 0\}$. Since $L$ is non-regular, $\overline{L}$ (the complement of $L$) is also nonregular. Yet, $L \cap \overline{L} = \emptyset$ is regular.

2.3 Answer: True. We proved in class that if a language is regular, so is its complement. This implies the above statement.

2.4 Answer: False. For a counterexample, let $A_k = \{0^k1^k\}$. Clearly, for all $k > 0$, since $A_k$ is a finite language, it is regular. However, $A_1 \cup A_2 \cup A_3 \cup \ldots = \{0^k1^k \mid k > 0\}$, which is nonregular.

2.5 (Note: A string $w$ is in the intersection if and only if it is in every $A_i$.)

Answer: False. For a counterexample, let $S = \{0^p \mid p \text{ is a prime }\}$. Define $A_k$ as $0^* - \{0^x \mid x \text{ is the } k\text{th nonprime}\}$. Clearly, each $A_k$ is regular, yet $A_1 \cap A_2 \cap A_3 \cap \ldots$ is $S$, which is nonregular.

3

Answer: Here is the intuition for how to construct a machine $M'$ for MAX($L$): If $q_f$ is a final state of $M$ and there is a non-empty string that drives $M$ from $q_f$ to a final state (possibly $q_f$ itself), then $q_f$ should not be a final state in $M'$. This ensures that $M'$ does not accept a string in $L$ if there is a way of extending it to be another string in $L$.

Formally, let DFA $M' = (Q', \Sigma, \delta', q'_0, F')$, where:

$Q' = Q$
$q'_0 = q_0$
$\delta' = \delta$

$F' = \{q \mid q \in F \text{ and for all non-empty strings } y, \delta(q,y) \not\in F\}$

Clearly $M'$ recognizes MAX($L$).

4

Answer: We observe that a string $w$ is in CYCLE($L$) if and only if there is a way of dividing $w$ into two parts $x_1$ and $x_2$ and there is a state $q$ of the original machine such that a marble starting off in state $q$ ends up in a final state of $M$ upon consuming $x_1$ and a marble starting off in the initial state of $M$ ends up in $q$ upon consuming $x_2$. This suggests that the marble should keep track of three things: (i) the state of $M$ where it started, (ii) which state of $M$ it currently is, and (iii) whether it is still consuming the
Formally, let NFA \( M' = (Q', \Sigma, \delta', q_0', F') \), where:
\[
Q' = \{ q_0' \} \cup \{ [p, q, i] \mid p, q \in Q, i \in \{ 1, 2 \} \}
\]
\[
F' = \{ [q, q, 2] \mid q \in Q \}
\]
\[
\delta'(q_0', \varepsilon) = \{ [p, p, 1] \mid p \in Q \}
\]
For all \( p, q \in Q, i \in \{ 1, 2 \}, a \in \Sigma, \)
\[
\delta'([p, q, i], a) = \{ [p, \delta(q, a), i] \}
\]
For all \( q \in F, p \in Q, \)
\[
\delta'([p, q, 1], \varepsilon) = \{ [p, q, 0, 2] \}
\]

5

5.1 \( L = \{ 0^m1^n0^{m+n} \mid m, n \text{ are any natural numbers} \} \)

Answer: \( L \) is not regular. For a proof, assume \( L \) is regular. Let \( s = 0^p1^p0^{2p} \), where \( p \) is the constant mentioned in PL. Clearly \( s \) is in \( L \) and \( |s| \geq p \). So, by PL, there are \( x, y, z \) such that \( s = xyz, |xy| \leq p, |y| \geq 1, xy'z \) is in \( L \) for all \( i \geq 0 \). Since \( |xy| \leq p, y \) lies entirely in the first sequence of 0’s. Hence, \( xy^2z = 0^{p+|y|1^p0^{2p}} \). Since \( |y| \geq 1 \), it follows that \( xy^2z \) is not in \( L \), contradicting PL. We conclude that \( L \) is regular.

5.2 \( L = \{ 0^m1^n \mid m \text{ divides } n \} \)

Answer: \( L \) is not regular. For a proof, assume \( L \) is regular. Let \( p \) be the constant mentioned in PL. Let \( s = 0^q1^q \), where \( q \) is a prime number greater than \( p + 1 \). Clearly \( s \) is in \( L \) and \( |s| \geq p \). So, by PL, there are \( x, y, z \) such that \( s = xyz, |xy| \leq p, |y| \geq 1, xy'z \) is in \( L \) for all \( i \geq 0 \). Since \( |xy| \leq p, y \) lies entirely in the first sequence of 0’s. Hence, \( xy^0z = xz = 0^{p-|y|1^q} \). Since \( p \geq |y| \geq 1 \) and \( q \geq p + 2 \), it follows that \( q - 1 \geq q - |y| \geq 2 \). This, together with the fact that \( q \) is a prime, implies that \( q - |y| \) does not divide \( q \). Thus, \( xz \) is not in \( L \), contradicting PL. We conclude that \( L \) is not regular.

5.3 \( \{ xwx^R \mid x \text{ and } w \text{ are strings in } (0 \cup 1)^+ \} \)

Answer: This language is regular. To see this, note that a string is in \( L \) if and only if it begins and ends with the same symbol, and has at least three symbols. Thus, the following regular expression captures \( L \): \( 0(0 \cup 1)^+0 \cup 1(0 \cup 1)^+1 \). Since \( L \) has a regular expression, \( L \) is regular.

5.4 \( \{ 0^m \mid m = 2^n \text{ for some natural number } n \} \)

Answer: \( L \) is not regular. For a proof, assume \( L \) is regular. Let \( s = 0^{2^p} \), where \( p \) is the constant mentioned in PL. Clearly \( s \) is in \( L \) and \( |s| \geq p \). So, by Pumping Lemma, there are \( x, y, z \) such that \( s = xyz, |xy| \leq p, |y| \geq 1, xy'z \) is in \( L \) for all \( i \geq 0 \). We have:
\[
2^p < |xy^2z| \quad (\text{because } |xyz| = 2^p \text{ and } |y| \geq 1)
\]
\[ \leq 2^p + p \quad \text{(because } |xyz| = 2^p \text{ and } |xy| \leq p) \]

\[ < 2^p + 2^p \quad \text{(because } p < 2^p \text{ for any } p \geq 1) \]

\[ = 2^{p+1} \]

It follows that \(|xy^2z|\) is not a power of 2, and so \(xy^2z\) is not in \(L\). This contradicts PL. We conclude that \(L\) is not regular.

5.5 problem 1.28 in the book.

**Answer:** The language \(E\) is not regular. For a proof, assume \(E\) is regular. Let \(p\) be the constant mentioned in PL. Consider string \(s = \begin{bmatrix} 0 \\ 0 \\ 1^p \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \). Clearly \(s \in E\) and \(|s| \geq p\), therefore, by PL, there exist \(x, y, z\) such that \(s = xyz\), \(|xy| \leq p\), \(|y| \geq 1\), and \(xy^iz \in E\) for all \(i \geq 0\). Since \(|xy| \leq p\), it follows that \(y\) lies entirely in the first sequence of \(\begin{bmatrix} 0 \\ 0 \end{bmatrix}^i\). Therefore, we have \(xy^2z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{p+|y|} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^p\). Since \(|y| \geq 1\), it follows that \(xy^2z \not\in E\), contradicting PL. Hence, \(E\) is not regular.

5.6 \(\{0^m1^n \mid m \text{ is not equal to } n\}\)

**Answer:** \(L\) is not regular. For a proof, let \(A\) denote the language \(\{0^n1^n \mid n \geq 0\}\). We observe that \(A = R \cup \overline{L}\), where \(R\) is the complement of \(0^*1^*\). Since \(0^*1^*\) is regular, its complement, namely, \(R\) is regular. If \(L\) were regular, then since regular languages are closed under union, \(R \cup L\) would be regular. Since regular languages are closed under complementation, it follows that \(R \cup \overline{L}\), which is \(A\), is regular, contradicting the well known fact that \(A\) is not regular. We conclude that \(L\) is not regular.