Please think carefully about how you are going to organise your answers before you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Do both parts of Exercise 1.21 from the textbook. Use the \( R_{ij}^k \) method as described in the lecture notes on the course website; do not use the textbook’s “GNFA method.” You may use the shorthand \( R^+ \) to denote \( RR^* \), where \( R \) is an arbitrary regular expression. Try to simplify the intermediate regular expressions, so as to make your own life simple! \([6+7\text{ points}]\)

2. For a language \( L \) over alphabet \( \Sigma \), define \( \text{MAX}(L) = \{ x \in L : x \text{ is not a proper prefix of any string in } L \} \). Recall that \( y \) is said to be a proper prefix of \( x \) if \( y \) is a prefix of \( x \) and \( y \neq x \). Prove that if \( L \) is regular, then so is \( \text{MAX}(L) \). \([8\text{ points}]\)

3. For a language \( L \) over alphabet \( \Sigma \), define \( \text{CYCLE}(L) = \{ xy : x, y \in \Sigma^* \text{ and } yx \in L \} \). Prove that if \( L \) is regular, then so is \( \text{CYCLE}(L) \). \([12\text{ points}]\)

4. For each of the following languages, say whether or not the language is regular and prove your answer. To prove that a language is regular, specify a finite automaton or a regular expression for that language. To prove that a language is not regular, use the pumping lemma or closure properties of regular languages. Proofs must be precisely written. Make sure you fully understand the definitions of the sets before answering.

4.1. \( \{0^m1^n0^{m+n} : m, n \geq 0\} \). \([7\text{ points}]\)

4.2. \( \{0^m1^n : m \text{ divides } n\} \). \([7\text{ points}]\)

4.3. \( \{xwx^R : x, w \in \{0, 1\}^*, |x| > 0 \text{ and } |w| > 0\} \). \([7\text{ points}]\)

4.4. \( \{0^2^n : n \geq 0\} \). \([7\text{ points}]\)

4.5. Problem 1.35 from the textbook. \([7\text{ points}]\)

4.6. \( \{0^m1^n : m, n \geq 0 \text{ and } m \neq n\} \). \([7\text{ points}]\)

5. Are the following statements always true? If true, give a brief justification and if false, give a concrete counterexample. Below, \( A \) and \( B \) denote languages over some alphabet \( \Sigma \).

5.1. If \( A \cup B \) is regular, then at least one of \( A \) and \( B \) is regular. \([5\text{ points}]\)

5.2. If \( A \cap B \) is regular, then at least one of \( A \) and \( B \) is regular. \([5\text{ points}]\)
5.3. If $\overline{A}$ (defined as $\Sigma^* - A$) is regular, then $A$ is regular. [5 points]

5.4. A union of arbitrarily many regular languages is regular, even if it is an infinite union. [5 points]

5.5. An intersection of arbitrarily many regular languages is regular, even if it is an infinite intersection. [5 points]

Challenge Problems

CP3: Let $L$ be any subset of $0^*$. Prove that $L^*$ is regular.

This is a delightful problem and will teach you something nice about regular languages if you solve it.