Please think carefully about how you are going to organise your answers before you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Pick any one of the following two subparts and turn it in.

[OPTION 1] Look at Exercise 1.21 in the textbook. Convert the two DFAs in that exercise to regular expressions, but do not use the textbook’s “GNFA method”. Instead, use the $R_{kj}$ method discussed in class, and described in the lecture notes (on the course website). You may use the shorthand $X^+$ to denote $XX^*$, where $X$ is an arbitrary regular expression. Try to simplify the intermediate regular expressions; this will save you from a lot of pain. [6+7 points]

[OPTION 2] Give a full formal proof that the subset construction, for converting an NFA into an equivalent DFA, actually works. Make every step mathematically precise, using only the formal definitions of acceptance for a DFA and an NFA, and the formal definition of the $\varepsilon$-closure function $E(\cdot)$. You may use informal wording like “following arrows” to add explanation to your proof, but not in lieu of a formal proof. [13 points]

2. For a language $L$ over alphabet $\Sigma$, define $\text{MAX}(L) = \{x \in L : x \text{ is not a proper prefix of any string in } L\}$. Recall that $y$ is said to be a proper prefix of $x$ if $y$ is a prefix of $x$ and $y \neq x$. Prove that if $L$ is regular, then so is $\text{MAX}(L)$. [8 points]

3. For a language $L$ over alphabet $\Sigma$, define $\text{CYCLE}(L) = \{xy : x, y \in \Sigma^* \text{ and } yx \in L\}$. Prove that if $L$ is regular, then so is $\text{CYCLE}(L)$. [12 points]

4. For each of the following languages, say whether or not the language is regular and prove your answer. To prove that a language is regular, specify a finite automaton or a regular expression for that language. To prove that a language is not regular, use the pumping lemma or closure properties of regular languages.

Proofs must be precisely written. Make sure you fully understand the definitions of the sets before answering.

4.1. $\{0^m1^n0^{m+n} : m, n \geq 0\}$. [7 points]

4.2. $\{0^m1^n : m \text{ divides } n\}$. [7 points]

4.3. $\{xwx^R : x, w \in \{0,1\}^*, |x| > 0 \text{ and } |w| > 0\}$. [7 points]

4.4. $\{0^n : n \geq 0\}$. [7 points]

4.5. Problem 1.35 from the textbook. [7 points]

4.6. $\{0^m1^n : m, n \geq 0 \text{ and } m \neq n\}$. [7 points]
5. Are the following statements always true? If true, give a brief justification and if false, give a concrete counterexample. Below, $A$ and $B$ denote languages over some alphabet $\Sigma$.

5.1. If $A \cup B$ is regular, then at least one of $A$ and $B$ is regular. [5 points]

5.2. If $A \cap B$ is regular, then at least one of $A$ and $B$ is regular. [5 points]

5.3. If $\overline{A}$ (defined as $\Sigma^* - A$) is regular, then $A$ is regular. [5 points]

5.4. A union of arbitrarily many regular languages is regular, even if it is an infinite union. [5 points]

5.5. An intersection of arbitrarily many regular languages is regular, even if it is an infinite intersection. [5 points]

Challenge Problems

**CP3:** Let $L$ be any subset of $0^*$. Prove that $L^*$ is regular.

This is a delightful problem and will teach you something nice about regular languages if you solve it.