Homework 4
Due 2017-Feb-06, 11:59pm

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General Instructions. Same as for Homework 1.

Honor Prinicple. Same as for Homework 1.

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8. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$, let $\mathrm{R}_\varepsilon(f)$ denote its ε-error (two-sided) randomized query complexity. That is, we are considering distributions over decision trees with a *worst case* bound on the number of queries made, but we allow the trees to make errors. Formally, for a probability distribution τ on n-input binary decision trees, let $\mathrm{depth}(\tau)$ denote the *maximum* depth of a tree in the support of τ. Let $T(x)$ denote the output of the decision tree T on the input x. Then
$$\mathrm{R}_\varepsilon(f) \;=\; \min\{\mathrm{depth}(\tau) : \forall x \in \{0,1\}^n \Pr_{T\sim\tau}[T(x) \neq f(x)] \le \varepsilon\}\,.$$
Let $\mathrm{R}(f)$ denote the zero-error version of randomized query complexity (i.e., the "Las Vegas" version):
$$\mathrm{R}_0(f) \;=\; \min_{\tau} \max_{x\in\{0,1\}^n} \mathbb{E}_{T\sim\tau}[\mathrm{cost}(T,x)]\,,$$
where the min is over all distributions over trees that *always* correctly compute $f(x)$.

   This problem walks you through some general theorems about randomized query complexity, and its relation to other measures.

   8.1. Show that $\mathrm{R}_\varepsilon(f) \le \varepsilon^{-1}\,\mathrm{R}_0(f)$. In particular, $\mathrm{R}_{1/3}(f) = O(\mathrm{R}_0(f))$. There is an easy proof via Markov's inequality from probability theory.

   8.2. Prove that $\mathrm{R}_\varepsilon(f) \ge (1-2\varepsilon)\,\mathrm{bs}(f)$. For this, consider a distribution τ that achieves the minimum in the definition of R_ε and then consider the probability that a random tree from τ queries each sensitive block of an input.

   8.3. Let $\widetilde{\deg}_\varepsilon(f)$ denote the ε-approximate degree of f, i.e.,
$$\widetilde{\deg}_\varepsilon(f) \;=\; \min\{\deg(p) : p \text{ multilinear polynomial and } \forall\, x \in \{0,1\}^n \text{ we have } |p(x) - f(x)| \le \varepsilon\}\,.$$
Let $T \sim \tau$, where τ is a distribution over depth-d decision trees. Let $x \in \{0,1\}^n$. Show that $\Pr_T[T(x) = 1]$ can be expressed as a degree-d polynomial in $x_1,\ldots,x_n$. Using this, prove that $\widetilde{\deg}_\varepsilon(f) \le \mathrm{R}_\varepsilon(f)$ for $\varepsilon \in (0,\frac{1}{3})$.

   8.4. Prove that $\mathrm{bs}(f) \le 6\,\widetilde{\deg}_{1/3}(f)^2$. Generalize the argument from class (using Markov's inequality from polynomial approximation theory) relating $\mathrm{bs}(f)$ to $\deg(f)$. Conclude that $\widetilde{\deg}_{1/3}(f)$ and $\mathrm{R}_{1/3}(f)$ are polynomially related.

9. Treating an n-bit vector $\vec{x}$ as a *string* (so that the ordering of the variables *is* important) we can define a Boolean function $A_n^s : \{0,1\}^n \to \{0,1\}$ as follows: $A_n^s(\vec{x}) = 1$ iff $\vec{x}$ contains s as a substring. Here s is a fixed string.

   9.1. Show that A_5^{111} and A_6^{111} are not evasive, i.e., it is possible to compute these functions without ever having to look at every input bit.

   9.2. Show that A_3^{111} and A_4^{111} are evasive.

   9.3. Prove that A_n^{111} is evasive iff $n \equiv 0$ or $3 \pmod 4$.
   Hint: For the "if" direction, use a recurrence for the number of strings of length n that satisfy A_n^{111}, and prove that sometimes just looking at this number tells you that the function is evasive. For the "only if" direction, use the ideas from your solution to #9.1, plus induction.

   9.4. Find all integers n for which A_n^{100} is evasive.
   Hint: Consider n mod 3.