Lecture 9: Sparsest Cut and Metric Embeddings

In the sparsest cut problem the input is an undirected graph $G = (V, E)$. Each edge has a quantity $c(e)$ which we now think of as the capacity of the cut. There also exists demand pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$, and pair i has a demand D_i. Given a subset of vertices $S \subseteq V$, let $\texttt{sep}(S) := \{i : |(s_i, t_i) \cap S| = 1\}$, and let $D(S) := \sum_{i \in S} D_i$, and the *sparsity* of S is defined as

$$\Phi(S) := \frac{c(\delta(S))}{D(\texttt{sep}(S))}$$

The sparsest cut problem is to find the cut of minimum sparsity, and

$$\Phi^* := \min_{S \subseteq V} \frac{c(\delta(S))}{D(\texttt{sep}(S))}$$

Given a set $S \subseteq V$, we associate the distance $d_S(u, v) := 1$ if exactly one of u and v are in S, and 0 otherwise. Such a distance function is called an elementary cut metric on V. It is straightforward to see that

$$\Phi^* = \min_{d\text{:elementary cut metric}} \frac{\sum_{(u,v) \in E} c(u, v) d(u, v)}{\sum_{i=1}^{k} D_i d(s_i, t_i)} \tag{1}$$

To get a lower bound on the sparsity, we minimize over *general metrics* instead of elementary cut metrics. This, as we saw last time, can be done in polynomial time via the following LP.

$$\min \quad \sum_{(u,v)} c(u, v) d(u, v) \qquad d(u, v) \in [0, 1] \tag{2}$$

$$d(u, v) \leq d(u, w) + d(w, v) \qquad \forall (u, v, w) \in V \times V \times V \tag{3}$$

$$\sum_{i=1}^{k} D_i d(s_i, t_i) \geq 1 \tag{4}$$

We first show how to use the multicut algorithm done last time to obtain an approximation for the sparsest cut problem. Subsequently, we will see how the theory of metric embeddings will help give a better approximation.

Sparsest Cut from Multicut

Let $D := \sum_{i=1}^{k} D_i$. Solve (2) to get distances $d(\cdot, \cdot)$.

Claim 1. *We can find a set of demand pairs* $S \subseteq [k]$ *such that*

$$d(s_i, t_i) \geq \frac{1}{H(D) \sum_{i \in S} D_i}$$

for all $i \in S$*, where* $H(\ell)$ *is the harmonic number* $1 + 1/2 + \cdots + 1/\ell$*.*

Proof. Rename the demand pairs such that $d(s_1, t_1) \geq \cdots \geq d(s_k, t_k)$. Let S_i be the set $\{1, 2, \ldots, i\}$. We show that one of these S_i's will satisfy the conditions of the claim. Suppose not. That is, $d(s_i, t_i) < \frac{1}{H(D)D(S_i)}$. Note that $D_i := D(S_i) - D(S_{i-1})$ (with $D(\emptyset) := 0$). So, we get

$$\sum_{i=1}^{k} D_i d(s_i, t_i) < \frac{1}{H(D)} \sum_{i=1}^{k} \frac{D(S_i) - D(S_{i-1})}{D(S_i)} \leq \frac{1}{H(D)} \left(1 + \frac{1}{2} + \cdots + \frac{1}{D}\right) = 1$$

which contradicts (4). □

Let S be the set with $d(s_i, t_i) \geq \frac{1}{H(D)D(S)}$. We now define $d'(u, v) := D(S)H(D) \cdot d(u, v)$ for all u, v. Note that d' remains a distance (satisfies triangle inequality) and $d'(s_i, t_i) \geq 1$ for all $i \in S$; that is, d' is a feasible solution to the multicut LP when the terminals are the set S. This gives us the existence of a set of edges $F \subseteq E$ such that each pair s_i is separated from t_i and $c(F) = O(\ln |S|) \cdot \sum_{u,v} c(u, v) d'(u, v)$.

Suppose F separates the graph into pieces $X_1, X_2, \ldots, X_r$. We claim that one of the X_i's will have low sparsity. Note that $F = \bigcup_i \delta X_i$, and each edge of F is in at most two δX_i's. So,

$$\sum_{i=1}^{r} c(\delta X_i) \leq 2c(F)$$

Furthermore, $\bigcup_i \mathtt{sep}(X_i)$ contains all demand pairs in S, and each demand pair (s_i, t_i) lies in exactly two $\mathtt{sep}(X_j)$'s. So,

$$\sum_{i=1}^{r} D(\mathtt{sep}(X_i)) = 2D(S)$$

Therefore,

$$\min_{i=1}^{r} \frac{c(\delta(X_i))}{D(\mathtt{sep}(X_i))} \leq \frac{\sum_{i=1}^{r} c(\delta(X_i))}{\sum_{i=1}^{r} D(\mathtt{sep}(X_i))} \leq \frac{2c(F)}{2D(S)} = \frac{O(\ln k) D(S) H(D) \mathtt{lp}}{D(S)} = O(\log k \log D) \mathtt{lp}$$

giving us a $O(\log k \log D)$-approximation.

Sparsest Cut from Metric Embeddings

Let's recall the definition of sparsity

$$\Phi^* = \min_{d:\text{elementary cut metric}} \frac{\sum_{(u,v) \in E} c(u, v) d(u, v)}{\sum_{i=1}^{k} D_i d(s_i, t_i)} \tag{5}$$

A metric d on V is called a cut metric if it is a linear combination of elementary cut metrics. The set of all such metrics is the cone of all elementary cut metrics on V.

$$CUT := \{d : \exists \lambda_S : \quad d = \sum_{S \subseteq V} \lambda_S d_S\}$$

The following claim shows that in the sparsity definition we could minimize over all cut metrics instead of elementary cut metrics.

Claim 2.

$$\Phi^* = \min_{d \in CUT} \frac{\sum_{(u,v)\in E} c(u,v) d(u,v)}{\sum_{i=1}^k D_i d(s_i, t_i)} \tag{6}$$

Proof. Since we are minimizing over a larger space, we have $\Phi^* \geq RHS$. Let d be the minimizer of the RHS. Since, $d = \sum_S \lambda_S d_S$, we get

$$\begin{aligned} RHS &= \frac{\sum_{(u,v)\in E} c(u,v) \sum_{S\subseteq V} \lambda_S d_S(u,v)}{\sum_{i=1}^k D_i \sum_{S\subseteq V} \lambda_S d_S(s_i,t_i)} \\ &= \frac{\sum_{S\subseteq V} \lambda_S \sum_{(u,v)\in E} c(u,v) d_S(u,v)}{\sum_{S\subseteq V} \lambda_S \sum_{i=1}^k D_i d_S(s_i,t_i)} \\ &= \frac{\sum_{S\subseteq V} \lambda_S c(\delta(S))}{\sum_{S\subseteq V} \lambda_S D(S)} \\ &\geq \min_{S\subseteq V : \lambda_S > 0} \frac{c(\delta(S))}{D(S)} \geq \Phi^* \end{aligned}$$

where the last but one inequality used the elementary fact that for positive reals $a_1, \ldots, a_t, b_1, \ldots, b_t$, we have $(\sum_{i=1}^t a_i)/(\sum_{i=1}^t b_i) \geq \min_{i=1}^t (a_i/b_i)$. □

A metric d on V is called an $\mathcal{L}_1$ metric if there is a mapping $\phi : V \to \mathbb{R}^h$ for some h such that $d(u,v)$ is the ℓ_1 distance between $\phi(u)$ and $\phi(v)$, that is,

$$d(u,v) = ||\phi(u) - \phi(v)||_1 = \sum_{i=1}^h |\phi(u)(i) - \phi(v)(i)|$$

Lemma 1. *For any (V,d) with $d \in CUT$, there is a mapping $\phi : V \to \mathbb{R}^h$ such that $||\phi(u)-\phi(v)||_1 = d(u,v)$ for all pairs for some $h > 0$. Conversely, given a mapping $\phi : V \to R^h$, there exists $d \in CUT$ such that $d(u,v) = ||\phi(u)-\phi(v)||_1$ for all pairs u and v. The second mapping is polytime computable and has $\lambda_S > 0$ for at most nh sets.*

Proof. If $d = \sum_S \lambda_S d_S$, define a mapping on h coordinates where $h = |\{S : \lambda_S > 0\}|$, as follows. $\phi(u)(S) := \lambda_S \cdot \mathbf{1}_{u\in S}$. Note that $||\phi(u) - \phi(v)||_1 = d(u,v)$ for any pair u and v.

For the converse, we have h sets for each coordinate. Fix a coordinate i and order the vertices as $\phi(u_1)(i) \leq \phi(u_2)(i) \leq \cdots$. The sets with positive S are precisely $\{u_1, \ldots, u_t\}$ for $t = 1..h$. $\lambda_S = \phi(u_t)(i) - \phi(u_{t-1})(i)$ for $S = \{u_1, \ldots, u_t\}$ with $\phi(u_0)(i)$ defined as 0. Check that $d(u,v) = ||\phi(u) - \phi(v)||_1$. □

Thus, we get

$$\Phi^* := \min_{d \in \mathcal{L}_1} \frac{\sum_{(u,v)\in E} c(u,v)d(u,v)}{\sum_{i=1}^k D_i d(s_i,t_i)}$$

To go ahead, we define the notion of metric embedding. Given two metric spaces (V,d) and (V',d'), we call a mapping $\phi : V \to V'$ an embedding if it is one-to-one. The mapping has *dilation* at most $\alpha \geq 1$ and contraction at most $\beta > 1$ if for any pair of vertices u, v in V, we have

$$\frac{d(u,v)}{\alpha} \leq d'(\phi(u),\phi(v)) \leq \beta \cdot d(u,v)$$

The distortion of the mapping is the quantity $\rho = \alpha\beta$.

The following strong theorem of Bourgain shows that *metric* can be embedded into $\mathcal{L}_1$ with $O(\log n)$ distortion. We will prove this in the next class.

Theorem 1. *Given any metric space (V,d), there is a mapping $\phi : V \to \mathbb{R}^{O(\log^2 n)}$ such that with high probability, we have that for any pair of vertices u and v, $\frac{d(u,v)}{O(\log n)} \leq ||\phi(u) - \phi(v)||_1 \leq d(u,v)$.*

Now we are ready to get a $O(\log n)$ approximation for the sparsest cut problem. First solve (2) cut to get a "general" metric d on the vertices with $\sum_{i=1}^k D_i d(s_i,t_i) \geq 1$ and $\sum_{(u,v)} c(u,v)d(u,v) \leq$ `lp`. Use Bourgain's theorem to get the mapping $\phi : V \to \mathbb{R}^{O(\log^2 n)}$ with the property in the theorem. Then, use Lemma 1 to get cut-metric d' with $d'(u,v) = ||\phi(u) - \phi(v)||_1$; also obtain the decomposition into elementary cut-metrics. So, we have for any pair u, v, $d(u,v) \leq O(\log n)d'(u,v)$ and $d'(u,v) \leq d(u,v)$. Choose the set S with $\lambda_S > 0$ of minimum $\Phi(S)$.

$$\Phi(S) \leq \frac{\sum_{(u,v)} c(u,v)d'(u,v)}{\sum_i D_i d'(s_i,t_i)} \leq O(\log n)\frac{\sum_{(u,v)} c(u,v)d(u,v)}{\sum_i D_i d'(s_i,t_i)} \leq O(\log n)\texttt{lp}$$

where the numerator used $d'(u,v) \leq d(u,v)$, and the denominator used $d(u,v) \leq O(\log n)d'(u,v)$.

In fact, Bourgain's proof can be modified to show the following

Theorem 2. *Given any metric space (V,d) and a set $S \subseteq V$ of size at most k, there is a mapping $\phi : V \to \mathbb{R}^{O(\log^2 k)}$ such that with high probability, we have that for any pair of vertices u and v, $||\phi(u) - \phi(v)||_1 \leq d(u,v)$ and for any pair $u, v \in S$, $d(u,v) \leq O(\log k)||\phi(u) - \phi(v)||_1$.*