Problems in Approximation Algorithms

CIS800

Due: November 11th, 2010

Exercise 1. (a) Recall the LP relaxation for the set cover problem done in class. We showed that the integrality gap of the LP is at most f, where f is the maximum frequency of an element. For all f, construct an instance of set cover where the frequencies are bounded by f and $\texttt{opt} = f \cdot \texttt{lp}$.

(b) Construct an example of a set cover with n elements such that $\texttt{opt} = \Omega(\log n) \cdot \texttt{lp}$.
Hint: Consider a set system with k sets and $\binom{k}{k/2}$ elements with each element in $k/2$ sets.

Exercise 2. Consider the knapsack problem: we are given a knapsack of size B and n single copy items with item j having profit p_j and weight w_j. Assume $w_j \leq B$ for all j. The goal is to pick a subset of items which fits into the knapsack and gives maximum profit. The LP relaxation is as follows:

$$\max\{\sum_{j=1}^{n} p_j x_j : \quad \sum_{j=1}^{n} w_j x_j \leq B; \quad 0 \leq x_j \leq 1, \ \forall j \in [n]\}$$

Let x be a basic feasible solution of the above LP.

(a) Let $F := \{j : 0 < x_j < 1\}$. What can you say about $|F|$?

(b) Can you use the above to get a 1/2-approximation for the knapsack problem?

(c) What is the integrality gap of the above LP?

(d) Can you get a better approximation if we have the guarantee that $p_j \leq \varepsilon \cdot \texttt{opt}$ for all items j? Can you use this to get a $(1-\varepsilon)$ approximation in time $n^{O(1/\varepsilon)}$?
Hint: There cannot be more than $1/\varepsilon$ items having profit $p_j > \varepsilon \cdot \texttt{opt}$ in the final solution.

Exercise 3. Let's investigate LP relaxations for the max-cut problem. We'll think of a cut as a $\{0,1\}$ assignment on the vertices, and an edge is in the cut if its endpoints are assigned different values.

(a) (Undirected Graphs.) Let's have a variable x_{uv} for each edge (u, v) and a variable y_u for each vertex $u \in V$. Consider the following linear program

$$\texttt{lp}_U := \quad \max \sum_{(u,v)\in E} w_{uv} x_{uv} \qquad 0 \le x, y \le 1 \tag{1}$$
$$\text{subject to} \quad x_{uv} \le y_u + y_v \qquad \forall (u,v) \in E$$
$$x_{uv} \le 2 - (y_u + y_v) \qquad \forall (u,v) \in E$$

– Prove that (1) is a valid LP relaxation for the max-cut problem.
– Design an algorithm which returns a cut of weight at least $\texttt{lp}_U/2$.
– Show that the 2 above cannot be replaced by any smaller constant; that is the integrality gap of the LP is arbitrarily close to 1/2.

(b) (Directed Graphs.) As before, we have a variable x_{uv} for every directed edge (u, v) and y_u for every vertex. Consider the following linear program

$$\texttt{lp}_D := \quad \max \sum_{(u,v)\in E} w_{uv} x_{uv} \qquad 0 \le x, y \le 1 \tag{2}$$
$$\text{subject to} \quad x_{uv} \le y_v \qquad \forall (u,v) \in E$$
$$x_{uv} \le 1 - y_u \qquad \forall (u,v) \in E$$

– Prove that (2) is a valid LP relaxation for the max-cut problem in directed graphs.
– What's the best upper and lower bounds you can prove on the integrality gap of this LP?

Hint: You might want to recall that we already have local search algorithms which return cuts of weight $w(E)/2$ and $w(E)/4$ for the two cases respectively.

Exercise 4. Consider the following problem called maximum budgeted allocation (MBA). There are m items and n agents. Each agent i bids b_{ij} on item j, and has a budget B_i. On getting a subset S of items, agent i pays $\min\left(B_i, \sum_{j\in S} b_{ij}\right)$. The problems is to find an allocation of items to agents which generates the maximum revenue.

Cast this problem as that of maximizing a submodular function over a matroid constraint and argue there exists a randomized $(1-1/e)$ approximation algorithm for the problem. Check whether the two oracles can be simulated in polynomial time. 1. *Value oracle*: given set S, return $f(S)$. 2. *Independence Oracle*: given set S, return in $S \in \mathcal{I}$ or not.
Hint: Think partition matroids to capture that an item can go to at most one agent.

Exercise 5. Recall the definition of a weakly supermodular function r. For any two subsets $A, B \subseteq V$, at least one of the two holds

1. $r(A \cup B) + r(A \cap B) \ge r(A) + r(B)$
2. $r(A \setminus B) + r(B \setminus A) \ge r(A) + r(B)$.

Given an edge e, define the residual function $r'(S)$ as follows: $r'(S) = r(S)$ if $e \notin \delta(S)$; otherwise $r'(S) = r(S) - 1$. Prove that r' is also weakly supermodular.