E0234 Randomized Algorithms

Endterm

27th Apr, 2016

Good luck!

1. (**10 points.**) Suppose we get k i.i.d. samples $X_1, \dots, X_k$ from a distribution with mean μ and std dev σ, both of which are *unknown*. We have seen that $Z := \frac{1}{k}\sum_{i=1}^{k} X_i$ is an unbiased estimator of μ. Describe an unbiased estimator of the variance, σ^2.

2. (**10 points**) A is an array of n distinct numbers. Consider the following algorithm to estimate the approximate median of A: sample k entries of A uniformly at random to get the set R and return r, the median of R. We wish to have the following guarantee:

$$\mathbf{Pr}\left[\mathsf{rank}(r) \in \left[\frac{n}{2} - \varepsilon n, \frac{n}{2} + \varepsilon n\right]\right] \geq 1 - \delta$$

where $\mathsf{rank}(r)$ is the index of r in the sorted array A. How big does k need to be in terms n, ε and δ?

3. (**15 points**)

   (a) (**5 points**) Let $X_0 = 0$, and for $j \geq 0$, let X_{j+1} be chosen uniformly from the real interval $[X_j, 1]$. Show that the sequence $Y_k = 2^k(1 - X_k)$ is a martingale.

   (b) (**10 points**) Suppose $G = G(n, dn)$ is a random graph with n vertices and dn edges chosen uniformly at random among all possible edges. Let MC_n denote the size of the maximum cut in G. Prove that with constant probability,

$$1 - \frac{4}{\sqrt{dn}} < \frac{MC_n}{\mathbf{E}[MC_n]} < 1 + \frac{4}{\sqrt{dn}}$$

4. (**10+5 points**) Let P be an undirected path starting from vertex 1 on the left to vertex n in the right. Given a permutation π of $\{1, 2, \dots, n\}$, orient the edges of P as follows: $(i, i+1)$ is oriented from left to right if $\pi(i) < \pi(i+1)$, and right to left otherwise. Let Z_π denote the length of the longest left-to-right directed path in this orientation of P.

(a) Find the largest k such that $\mathbf{Pr}[Z_\pi \geq k] \geq 1/2$, where the probability is over a uniformly chosen random permutation π. We are interested in the asymptotic relationship between k and n, if any.

(b) **Bonus 5 points.** How will you use the above fact to design an algorithm to find long paths in Hamiltonian graphs?

5. (**15 points**) Let G be an undirected, non-bipartite and connected graph with n vertices. Consider two independent random walks starting at two nodes u and v respectively. Show that the expected number of steps for the two walks to meet is $O(n^6)$.

6. (**20 points.**) Let A be an $n \times n$ matrix with $|A_{ij}| \leq 1$ entries. In class, we showed the existence of $x \in \{\pm 1\}^n$ such that $||Ax||_\infty \leq \sqrt{2n\ln(2n)}$. In this problem we see a better result in case A is row and column sparse. Suppose A has at most k non-zero entries in each row and column.

   (a) **(5 points)** Let x be a random $\{\pm 1\}^n$ vector. For any fixed row i, upper bound the probability that $|a_i^\top x| \geq \beta$ where a_i is the ith row of A.

   (b) **(5 points)** Prove there exists a ± 1 vector x with $||Ax||_\infty = O(\sqrt{k \ln n})$.

   (c) **(10 points)** Prove there exists a ± 1 vector x with $||Ax||_\infty = O(\sqrt{k \ln k})$. **Hint**: LLL.