E0234: Solution Sketch of Assignment 2

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In this assignment sheet (and life in general), you may find it useful to know Stirling's approximation which states

$$\text{For any integer } n, \qquad \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e \cdot n^{n+\frac{1}{2}} e^{-n}$$

where e and π are the famous constants.

1. Every evening, a man either visits his parents, who live northwards, or his friend, who lives southwards (but not both). In order to be fair, he goes to the bus stop every evening at a random time and takes either the northward or southward bus, whichever comes first. The two kinds of buses stop at the bus stop every 30 minutes with perfect regularity. Yet, he visits his parents only three times per month. Why?

   Solution sketch: Because the bus to parents is scheduled to come 3 minutes after the arrival of bus to friend ☺.

2. Let n be an even integer and assume $n > 10$. Let G_k be the (multi)-graph on n vertices formed by taking the union of k perfect matchings which are chosen uniformly at random from the set of all perfect matchings among n points. What is the smallest k for which G_k is connected with high probability? (Recall, whp implies that this probability should approach 1 as n tends to infinity.)

   Solution sketch: Let us call the resulting graph G_k. Consider $S \subset V$ of size $2r$ for $r \in \{1, 2, \ldots, \frac{n}{4}\}$. We call S bad if there are no edge crossing the cut S in G_k. Then we have following.

$$Pr[S \text{ bad }] = \left(\frac{2r-1}{n-1}\frac{2r-3}{n-3}\cdots\frac{1}{n-2r+1}\right)^k = \left(\frac{\binom{n/2}{r}}{\binom{n}{2r}}\right)^k$$

   Using union bound, we have the following.

$$Pr[\exists S bad] \leq \sum_{r=1}^{n/4}\binom{n}{2r}\left(\frac{\binom{n/2}{r}}{\binom{n}{2r}}\right)^k = \sum_{r=1}^{n/4}\frac{\binom{n/2}{r}^k}{\binom{n}{2r}^{k-1}} \leq \sum_{r=1}^{n/4}\frac{1}{\binom{n/2}{r}^{k-2}}$$

   The last inequality follows from Starling's approximation. We now choose $k = 3$. Then we have the following for a constant c.

$$Pr[\exists Sbad] \le \sum_{r=1}^{n/4} \frac{1}{\binom{n/2}{r}} \le \frac{2}{n} + \frac{n/4}{\binom{n/2}{2}} \le \frac{c}{n}$$

Now argue that for $k = 2$, G_k is disconnected with high probability.

3. n people queue up to attend a movie which has n seats. However, the first person has lost his ticket and sits in one of the empty seats uniformly at random. Subsequently, each person (and no one else has lost a ticket) sits either in his assigned seat, or if that seat is already taken sits in an empty seat uniformly at random. What is the expected number of people *not* sitting in their correct seats?

   Solution sketch: Let $T(n)$ denote the random variable denoting the number of people not sitting in their correct seat. Then we have the following recurrence:

$$T(n) = 1 + \frac{1}{n}\sum_{i=2}^{n} T(n-i+1)$$

   Solving we have $T(n) = H_n = \Theta(\log n)$

4. Given a permutation π of $\{1, 2, \ldots, n\}$, let $L(\pi)$ denote the length of the longest increasing subsequence in π. Note that a subsequence may not be contiguous; for instanct in the permutation $\pi = (1, 6, 4, 5, 2, 7, 3)$ for $n = 7$, the longest subsequence is $(1, 4, 5, 7)$ and so $L(\pi) = 4$. In this exercise, you need to prove $\mathbf{E}[L(\pi)] = \Theta(\sqrt{n})$.

   (a) Prove that $\mathbf{E}[L(\pi)] = O(\sqrt{n})$. **Hint:** For a fixed k, calculate (an upper bound) probability on the probability that $L(\pi) \ge k$.

   Solution sketch:
   $Pr[L(\pi) \ge k] \le \frac{\binom{n}{k}}{k!} \le \frac{n^k}{k^{2k+1}}$
   Now we have $\mathbf{E}[L(\pi)] = \sum_k Pr[L(\pi) \ge k] = \sum_{k=1}^{2e\sqrt{n}} Pr[L(\pi) \ge k] + \sum_{k=2e\sqrt{n}+1}^{n} Pr[L(\pi) \ge k] \le \sum_{k=1}^{2e\sqrt{n}} + \sum_{k=2e\sqrt{n}+1}^{n} Pr[L(\pi) \ge k] = O(\sqrt{n}) + \sum_{k=2e\sqrt{n}} Pr[L(\pi) \ge k] = O(\sqrt{n})$

$$\begin{aligned}
\mathbf{E}[L(\pi)] &= \sum_k Pr[L(\pi) \ge k] \\
&= \sum_{k=1}^{2e\sqrt{n}} Pr[L(\pi) \ge k] + \sum_{k=2e\sqrt{n}+1}^{n} Pr[L(\pi) \ge k] \\
&\le \sum_{k=1}^{2e\sqrt{n}} 1 + \sum_{k=2e\sqrt{n}+1}^{n} Pr[L(\pi) \ge k] \\
&= 2e\sqrt{n} + \sum_{k=2e\sqrt{n}}^{n} Pr[L(\pi) \ge k] \\
&\le O(\sqrt{n})
\end{aligned}$$

(b) Prove that $\mathbf{E}[L(\pi)] = \Omega(\sqrt{n})$. **Hint:** Assume n is a perfect square. For $i = 1, \ldots, \sqrt{n}$, define the indicator random variable X_i which takes the value 1 if and only if some entry in $(i-1)\sqrt{n}+1 \leq j \leq i\sqrt{n}$ satisfies $(i-1)\sqrt{n}+1 \leq \pi(j) \leq i\sqrt{n}$. Can you relate these $\sqrt{n}$ variables with $L(\pi)$?

Solution sketch: We have the following for any $i \in [\sqrt{n}]$.

$$Pr[X_i = 0] = \frac{\binom{n-\sqrt{n}}{\sqrt{n}}\sqrt{n}!(n-\sqrt{n})!}{n!} \leq \left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \leq \frac{1}{e}$$

Using linearity of expectation:

$$E\left[\sum_{i=1}^{\sqrt{n}} X_i\right] \geq \sqrt{n}\left(1 - \frac{1}{e}\right)$$