

E0234: Assignment 4

Due: Monday, 8th Feb 2016.

It is highly recommended you do not google for the answers to the questions below. You can discuss with your friends, but then mention that in your submission. The writing should solely be your own.

1. (Filling up the gaps from class presentation.)

- (a) For $X \sim \text{Poi}(\lambda)$, prove $\mathbf{Exp}[X] = \mathbf{Var}[X] = \lambda$. Given $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ which are independent, prove $X + Y \sim \text{Poi}(\lambda + \mu)$.
- (b) Let \mathcal{E} be an event in the balls-and-bins experiment whose probability is either monotonically increasing or decreasing with the number of balls thrown. Let \mathcal{E}' be the corresponding event when each bin independently has $Y_i \sim \text{Poi}(m/n)$ many balls. Prove that $\Pr[\mathcal{E}] \leq 2\Pr[\mathcal{E}']$. **Hint:** You can use without proof that for $X \sim \text{Poi}(\lambda)$, when λ is a positive integer, $\mathbf{Med}(X) = \lambda$.
- (c) Recall some notation from class: $Y_i \sim \text{Poi}(m/n)$ and $Z = \sum_i Y_i$. Let \mathcal{E} be an event which is monotonically decreasing with Z ; that is, $\Pr[\mathcal{E} | Z = k] \geq \Pr[\mathcal{E} | Z = k']$ if $k < k'$. Suppose $\Pr[\mathcal{E} | Z = a] - \Pr[\mathcal{E} | Z = b] \leq \delta$ for some non-negative integers $a < b$. Prove

$$\left| \Pr[\mathcal{E} | a \leq Z \leq b] - \Pr[\mathcal{E} | Z = c] \right| \leq \delta, \quad \forall a \leq c \leq b.$$

2. Consider the coupon collector setting. Let τ be the (random) time when you observe k copies of every coupon where k is a fixed integer > 1 . Prove that for any real constant c ,

$$\lim_{n \rightarrow \infty} \Pr[\tau > n(\ln n + (k-1) \ln \ln n + c)] = 1 - e^{-e^{-c}}$$

3. Consider n balls thrown into n bins. What can you say about the k th largest load? More precisely if the load profile is (X_1, \dots, X_n) , define $L_k := k$ th-largest (X_1, \dots, X_n) . Express the function $f(n, k)$ such that $L_k = \Theta(f(n, k))$ whp. Start with $k = 2$.
4. Consider a counter that receives packets arriving on a stream. The counter is initialized to 0. If the counter is currently at i and a new packet arrives, then the counter is incremented with probability 2^{-i} (and otherwise remains unchanged). Suppose n packets arrive in total, and let X_n be the random variable denoting the counter value at the end of the stream.

- (a) Show that $\mathbf{Exp}[2^{X_n}] = n + 1$.
- (b) Show that $\mathbf{Var}[2^{X_n}] = \frac{n(n-1)}{2}$.

Hint: Use induction on n .