E0234: Assignment 8

Due: Monday, March 21, 2016.

It is highly recommended you do not google for the answers to the questions below. You can discuss with your friends, but then mention that in your submission. The writing should solely be your own.

1. Show that if μ and ν are two distributions, $\|\mu - \nu\|_{\text{TV}} = \frac{1}{2}\|\mu - \nu\|_1$.

2. Do Exercise 11.6 of Motwani-Raghavan. You can assume that you have an algorithm that samples exactly from the uniform distribution of matchings in a graph.

3. Show that the Markov chain for C-coloring graphs of maximum degree Δ discussed in class is irreducible, if $C \geq \Delta + 2$. Moreover, prove that the stationary distribution of the Markov chain is the uniform distribution on C-colorings.

   Recall that each move in the Markov chain is to pick a random vertex v from the graph, a random color $c \in [C]$, and to color v with c if permitted and to otherwise leave the coloring unchanged.

4. Consider the following random walk on the hypercube $\{0,1\}^n$: with probability $1/(n+1)$, stay at current vertex; otherwise, with probability $1/(n+1)$ for each of the n neighbors, go to one of the neighbors. Note that the self-loop probability is $1/(n+1)$.

   An alternative way to view the walk is that for current state x, a random $i \in \{0, 1, \ldots, n\}$ is picked uniformly at random. If $i = 0$, x doesn't change; otherwise, x_i is flipped.

   Consider the following coupling (X_t, Y_t).

   - Suppose X_t and Y_t differ at only one coordinate i_0. Then, if X_t picks $i = 0$, Y_t picks i_0; if X_t picks i_0, then Y_t picks $i = 0$; else, both X_t and Y_t pick the same i.
   - Suppose X_t and Y_t differ at the subset of coordinates $S \subseteq [n]$, where $|S| > 1$. Fix a bijection $\pi : S \to S$ such that $\pi(i) \neq i$ for all $i \in S$. Then, if X_t picks $i = 0$, then Y_t also picks $i = 0$; if X_t picks $i \notin S$, then Y_t also picks i; if X_t picks $i \in S$, then Y_t picks $\pi(i)$.

   Observe that the distance between X_t and Y_t never increases. Analyze separately the expected time needed for the distance to decrease to 1 and then the expected time for the distance to go from 1 to 0. Use this to give a bound on the expected coupling time and, hence, the mixing time for this Markov chain.