E0234: Assignment 9

Due: Wednesday, March 30, 2016.

It is highly recommended you do not google for the answers to the questions below. You can discuss with your friends, but then mention that in your submission. The writing should solely be your own.

1. Show a strongly connected *directed* graph for which there exist vertices u and v such that the expected time to first reach v starting from u by a random walk is exponential in the number of vertices.

2. Consider a graph G known to be 3-colorable. We want to 2-color the graph such that no triangle is monochromatic (all three vertices have the same color).

   Consider the following algorithm. First, color G arbitrarily with red and blue. Then, as long as there is monochromatic triangle, pick a random vertex in that triangle and color it with the opposite color.

   Show that the algorithm terminates in polynomial time. To do this, fix a legal 3-coloring $\mathcal{C}$, and let dis be the number of vertices in which the current 2-coloring differs from $\mathcal{C}$. Analyze how dis changes as the algorithm proceeds, and argue that if the algorithm takes too long, dis can become 0.

3. Let $G_{n,N}$ be the uniform distribution on n-vertex graphs with exactly N edges. Suppose $N = cn$ for some constant $c > 0$. Let X be the expected number of isolated vertices (i.e., vertices of degree 0) for a random graph from $G_{n,N}$.

   (a) Determine $\mathbf{E}[X]$.

   (b) Using Azuma, show that $\Pr[|X - \mathbf{E}[X]| \geq 2\lambda\sqrt{cn}] \leq 2e^{-\lambda^2/2}$.

4. Consider a gambler playing a sequence of independent games, either winning one rupee with probability 1/2 or losing one rupee with probability 1/2. The gamber continues until either losing total of ℓ_1 or winning total of ℓ_2. Let X_i be +1 if he wins in the i'th game and −1 otherwise. Let $Z_t = (\sum_{i=1}^{t} X_i)^2 - t$.

   (a) Show that $Z_1, Z_2, \ldots$ form a martingale with respect to $X_1, X_2, \ldots$.

   (b) Let T be the stopping time when the gambler finishes playing. What is $\mathbf{E}[Z_T]$?

   (c) Calculate $\mathbf{E}[T]$. (*Hint*: Use what we showed in class about the probabilities of the player winner and losing at time T.)