E0234 Randomized Algorithms

Midterm: Solution Sketch

2nd March, 2016. 1:30pm to 4:30pm.

Good luck!

1. (a) Prove: for any two random variables X, Y, $\mathbf{Exp}[X + Y] = \mathbf{Exp}[X] + \mathbf{Exp}[Y]$.
   Use basic definitions.

   (b) Prove or disprove: for independent random variables X, Y, $\mathbf{Var}[XY] = \mathbf{Var}[X] \cdot \mathbf{Var}[Y]$.
   False. Take X and Y to be independent uniform random $0/1$ variables. $\mathbf{Var}[XY] = 3/16$ while $\mathbf{Var}[X] = \mathbf{Var}[Y] = 1/4$

2. Consider a set system where each set has exactly 10 elements and every element is present in exactly 10 sets. Can you colour the elements red or blue such that every set has elements of both colours?

   Yes. Use symmetric Lovász Local Lemma.

3. Consider the random graph $G_{n,p}$ when $p = cn^{-2/3}$, and let X be the number of 4-cliques in the graph.

   (a) What is $\mathbf{Exp}[X]$?
   $p^6\binom{n}{4} = \frac{c^6}{24} - o(1)$

   (b) What is an upper bound on $\mathbf{Var}[X]$?
   $\mathbf{Var}[X] \leq \frac{n^4}{24}p^6 + \frac{n^6}{8}p^11 + \frac{n^5}{6}p^9 \leq \frac{c^6}{24} + O(n^{-1})$

   (c) What is $\lim_{n\to\infty} \mathbf{Pr}[X = 0]$?
   Use Janson's inequality to get that the limit is $e^{-c^6/24}$.

4. Suppose you can draw independent samples of a real random variable X that has expectation 0 and standard deviation σ. Explain how to use only $O(\log n)$ samples from this source to generate a random variable Y with expectation μ such that $\mathbf{Pr}[|Y - \mu| > 2\sigma] < 1/n$.

   Let S be set of $O(\log n)$ samples from X, and let $S' = S \cup -S$. Expected value of the median of S' is 0, and the median deviates from 0 by more than 2σ with probability $1/n$. Finally, shift by μ to get desired variable.

5. Consider the following algorithm for the independent set problem on an n-node graph. Sample a random permutation σ of $\{1, 2, \ldots, n\}$. Initialize I to $\varnothing$. For $i = 1$ to n, place $\sigma(i)$ in I if it doesn't have an edge to any vertex in I. In expectation, how large an independent set do you pick?

   Expected size is $\sum_{i=1}^{n} \frac{1}{d_i+1}$ where d_i is the degree of i'th vertex. Follows from the fact that probability $\sigma(i)$ is less than $\min\{\sigma(j) : (i,j) \in E(G)\}$ equals $\frac{1}{d_i+1}$.

6. In d-dimensions, there can be at most d unit vectors which are orthogonal to each other. Call a pair of unit vectors ε-orthogonal if $|u^\top v| \le \varepsilon$. How large (in cardinality) a set of pairwise ε-orthogonal unit vectors can you construct in d-dimensions?

   Can be as large as $k = \exp(O(\varepsilon^2 d))$. Take a random unit vector $v_b = [\frac{b_1}{\sqrt{d}}, \frac{b_2}{\sqrt{d}}, \ldots, \frac{b_d}{\sqrt{d}}]$ where each b_i is $+1$ or -1 with equal probability. Then, for any two random b and b', $\mathbf{Exp}[v_b \cdot v_{b'}] = 0$ and $\mathbf{Pr}[|v_b \cdot v_{b'}| > \varepsilon] < \frac{1}{k^3}$ by the Chernoff bound. We can then do a union bound over $\binom{k}{2}$ pairs b and b' to get the claimed result.