E0234 Randomized Algorithms

Midterm

29th Feb, 2016. 1:30pm to 4:30pm.

Good luck!

1. (a) Prove: for any two random variables X, Y, $\mathbf{Exp}[X + Y] = \mathbf{Exp}[X] + \mathbf{Exp}[Y]$.

   (b) Prove or disprove: for independent random variables X, Y, $\mathbf{Var}[XY] = \mathbf{Var}[X] \cdot \mathbf{Var}[Y]$.

2. Consider a set system where each set has exactly 10 elements and every element is present in exactly 10 sets. Can you colour the elements red or blue such that every set has elements of both colours?

3. Consider the random graph $G_{n,p}$ when $p = cn^{-2/3}$, and let X be the number of 4-cliques in the graph.

   (a) What is $\mathbf{Exp}[X]$?

   (b) What is an upper bound on $\mathbf{Var}[X]$?

   (c) What is $\lim_{n\to\infty} \mathbf{Pr}[X = 0]$?

4. Suppose you can draw independent samples of a real random variable X that has expectation 0 and standard deviation σ. Explain how to use only $O(\log n)$ samples from this source to generate a random variable Y with expectation μ such that $\mathbf{Pr}[|Y - \mu| > 2\sigma] < 1/n$.

5. Consider the following algorithm for the independent set problem on an n-node graph. Sample a random permutation σ of $\{1, 2, \ldots, n\}$. Initialize I to $\emptyset$. For $i = 1$ to n, place $\sigma(i)$ in I if it doesn't have an edge to any vertex in I. In expectation, how large an independent set do you pick?

6. In d-dimensions, there can be at most d unit vectors which are orthogonal to each other. Call a pair of unit vectors ε-orthogonal if $|u^\intercal v| \le \varepsilon$. How large (in cardinality) a set of pairwise ε-orthogonal unit vectors can you construct in d-dimensions?