Lecture 17

Saturday, March 05, 2016
9:46 AM

- Last lecture, showed that approximate sampling from set of independent sets implies approx counting of # of independent sets
- Finish details of analysis
- Analysis extends to other problems with "self-reducibility"
    - See MR 11.3.1 for how to count perfect matchings assuming uniform sampler for perfect matchings
    - <u>Satisfiability</u>: Given SAT instance φ, approximately count # of satisfying assignments.

      For $\alpha \in \{0,1\}^*$, let $N_\alpha = \{ \beta \in \{0,1\}^n$ satisfying $\mid \beta_i = \alpha_i$ for $i \in [|\alpha|] \}$. Want to find $|N_\emptyset|$.

      Note that $\max\{|N_{\alpha 0}|, |N_{\alpha 1}|\} \geq \frac{1}{2}|N_\alpha|$.

      Then, $|N_\emptyset| = \frac{|N_\emptyset|}{|N_{\alpha_1}|} \cdot \frac{|N_{\alpha_1}|}{|N_{\alpha_2}|} \cdots \frac{|N_{\alpha_{n-1}}|}{|N_{\alpha_n}|}$

      where $|\alpha_i| = i$ and $\frac{|N_{\alpha_i}|}{|N_{\alpha_{i+1}}|} \geq \frac{1}{2}$. $(N_{\alpha_n} \neq \emptyset)$.

      Estimate each of these ratios by an approximately uniform sampler.
- An art to come up with Markov chains that mix fast.
- We'll look at 2 techniques to analyze mixing time:
  (1) Coupling (2) Canonical paths

<u>Coupling</u>

<u>Def</u>: For dists μ and ν on Ω, $\|\mu - \nu\|_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|$.

So, if μ and ν close in TV, then prob of any event occurring wrt μ or ν similar.

<u>Claim</u>: $\|\mu - \nu\|_{TV} = \frac{1}{2}\|\mu - \nu\|_1$ (Check!)

<u>Def</u>: For a MC with transition matrix P, define

$$\begin{cases} d(t) = \max_x \| P^t(x,\cdot) - \pi \|_{TV} \\ \bar{d}(t) = \max_{x,y} \| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV} \end{cases}$$

$d(t) \quad x$

$\bar{d}(t) = \max_{x,y} \| P^t(x, \cdot) - P^t(y, \cdot) \|_{TV}$

$t_{mix}(\varepsilon) = \min \{ t : d(t) \le \varepsilon \}$

$t_{mix} = t_{mix}(1/4)$.

<u>Lemma</u>: $t_{mix}(\varepsilon) \le t_{mix} \cdot \log(1/\varepsilon)$.

<u>Def</u>: A <u>coupling</u> between two distributions μ and ν is a pair of random variables (X, Y) such that $X \sim \mu$ and $Y \sim \nu$.

<u>Example</u>: Suppose $\mu = \nu$, uniform on $\{0, 1\}$.

Then, (1) (X, Y) where X and Y are independent samples from μ

(2) (X, X) where X is a sample from μ

are both couplings!

<u>Lemma</u>: $\|\mu - \nu\|_{TV} \le \Pr[X \ne Y]$

(In fact, an equality! An "optimal coupling" exists!)

<u>Pf</u>: For any $A \subseteq \Omega$,

$$\begin{aligned}
\mu(A) - \nu(A) &= \Pr[X \in A] - \Pr[Y \in A] \\
&\le \Pr[X \in A] - \Pr[X \in A, Y \in A] \\
&= \Pr[X \in A, Y \notin A] \\
&\le \Pr[X \ne Y]
\end{aligned}$$

$\therefore \|\mu - \nu\|_{TV} \le \Pr[X \ne Y]$.

- So, to show two distributions close in TV distance, suffices to construct coupling (X, Y) where $\Pr[X \ne Y]$ is small.

<u>Def</u>: For a MC, a coupling is a process $(X_t, Y_t)_{t=0}^{\infty}$ ... that (X_t) and (Y_t) both separately evolve ... different states

according to the MC but started at different states
We assume that once $X_t = Y_t$, $X_{t'} = Y_{t'}$ $\forall\ t' > t$.
(Simply run the chains together after t).

Key Lemma: Suppose (X_t, Y_t) is a coupling of a Markov Chain with $X_0 = x$ and $Y_0 = y$.

$$\tau_{couple} = \min\{t : X_t = Y_t\}$$

Then, $\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} \le \Pr[\tau_{couple} > t]$

Pf: $\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} = \Pr[X_t \neq Y_t]$
$= \Pr[\tau_{couple} > t]$

Corollary: $d(t) \le \bar{d}(t) \le \max_{x,y} \Pr[\tau_{couple} > t]$

Example: Lazy random walk on cycle: stay at current vertex w. prob. $\frac{1}{2}$, move left w. prob. $\frac{1}{4}$, move right w. prob. $\frac{1}{4}$.

Coupling: (X_t, Y_t) with $X_0 = x$, $Y_0 = y$. At every t, choose X_t w. prob. $\frac{1}{2}$ and move to left / right with equal prob. Otherwise, choose Y_t and move to left/right with equal prob. Once they meet, they stick together.

Note that they never cross over.

Let $D(t) = |X_t - Y_t|$. Increases or decreases by 1 w. prob. $1/2$ at each step.

Then: $\mathbb{E}[\min\{t : D_t = 0 \text{ or } D_t = n\}] \le n^2$

$$\Pr[\tau_{converge} > t] \le \frac{n^2}{t}$$

So, $d(t) \le \frac{n^2}{t}$ and $\tau_{mix} \le 4n^2$.

Example: Lazy random walk on hypercube: stay at current vertex w. prob. $\frac{1}{2}$ and move to a uniform random neighbor otherwise.

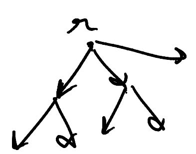
Equiv: Pick a random $i \in [n]$ and set x_i ...
bit.

Coupling: $X_0 = x$, $Y_0 = y$. At every $t \geq 0$, Pick a random $i \in [n]$ and update the i'th bit of both X_t and Y_t to the same random bit to get (X_{t+1}, Y_{t+1})

τ_{couple} is the time when all coordinates have been picked.

<u>Example</u>: Generating random spanning trees.

Consider arborescence rooted at r, all edges directed outwards from r.



Every vertex other than r has exactly one incoming edge.

MC: Choose a random edge (u, r). Add (u, r) to arborescence and remove the only incoming edge to u. Make u the new root.

<u>Coupling</u>: Evolve X_t and Y_t independently, until roots collide. After then, evolve them together.

$$\mathbb{E}[\tau_{couple}] = \mathbb{E}[\text{time for roots to meet}] + \mathbb{E}[\text{time for trees to become same} \mid \text{same roots}]$$

Time for two random walks to meet $= O(n^6)$

Cover time $= O(n^3)$

$$= \text{poly}(n)$$

<u>Example</u>: Generating random colorings.

Δ = max degree

<u>Fact</u>: Any graph with max deg Δ can be colored with $\Delta + 1$ colors.

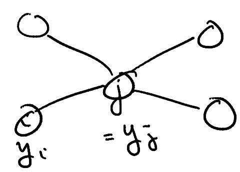
R.t. how to sample a random coloring? Can again be used to approx count # of colorings.

Consider following MC on C-colorings:

- Choose random vertex v
- Color v with random color $c \in [C]$ if permitted. Otherwise, don't change coloring.

<u>Lemma</u>: MC irreducible if $C \geq \Delta + 2$

<u>Pf</u>: Will show that there is a path between any two C-colorings x and y. Order vertices in some order and in that order, change x_i to y_i. May not be possible if blocked by future vertex j but can recolor if $C \geq \Delta + 2$.



<u>Lemma</u>: Stationary distribution is uniform. (Exercise!)

<u>Lemma</u>: Mixing time is $O(n \log n)$ if $C > 4\Delta + 1$.

<u>Pf</u>: Consider trivial coupling: both MC copies choose the same vertex and same color.

$$\Pr[d_{t+1} = d_t - 1 \mid d_t > 0] \geq \frac{d_t}{n} \cdot \frac{c - 2\Delta}{c}$$

$$\Pr[d_{t+1} = d_t + 1 \mid d_t > 0] \leq \frac{\Delta d_t}{n} \cdot \frac{2}{c}$$

$$\Rightarrow \mathbb{E}[d_{t+1} \mid d_t] \leq d_t \cdot \left(1 - \frac{c - 4\Delta}{cn}\right)$$

$$\Rightarrow \mathbb{E}[d_{t+1}] \leq \mathbb{E}[d_t] \cdot \left(1 - \frac{c - 4\Delta}{cn}\right)$$

$$\leq \mathbb{E}[d_t] \cdot \left(1 - \frac{1}{cn}\right)$$

$$\Rightarrow \Pr[d_t \geq 1] \leq n e^{-t/cn}$$ since $d_0 \leq n$.