Lecture 14

# Markov Chains

Monday, February 22, 2016
1:00 PM

- Sequence of random variables $X_0, X_1, \ldots X_t, \ldots$
- Mostly, we'll look at when the $X_t$'s are from a finite set $[n]$: finite Markov chains
- Key property: memorylessness

$$\Pr[X_t = a_t \mid X_{t-1} = a_{t-1}, \ldots, X_0 = a_0]$$
$$= \Pr[X_t = a_t \mid X_{t-1} = a_{t-1}]$$

- Not indep of history
- $X_0$ could be drawn from a distribution
- $P_{ij} = \Pr[X_t = j \mid X_{t-1} = i] \quad \forall t.$
- If $p_i(t) = \Pr[X_t = i]$ and $\bar{p}(t) = (p_1(t), \ldots, p_n(t))$,

  then $\bar{p}(t+1) = \bar{p}(t) \cdot P$

  $\Rightarrow \bar{p}(t) = \bar{p}(0) \cdot P^t$

- Useful rep as a wt'd directed graph
- Main object of study: $\bar{p}(\infty)$ and how fast it takes to get there.
- Example: 2SAT randomized algo [Pap 91]

  Given input $\varphi$:
  - Start with arbitrary assignment
  - Repeat upto $2cn^2$ times:
    - Choose an arbitrary unsat clause
    - Choose one of the two variables at

random and flip it

- If satisfying assignment found, return it. O'wise, declare unsatisfiable.

<u>Analysis</u>: Fix a satisfying assignment $A$, and let $X_t$ be number of variables in which assignment at step $t$ matches $A$.

$$\Pr[X_{t+1} = 1 \mid X_t = 0] = 1$$

$$\Pr[X_{t+1} = i+1 \mid X_t = i] \geq \frac{1}{2} \qquad \forall i > 0$$

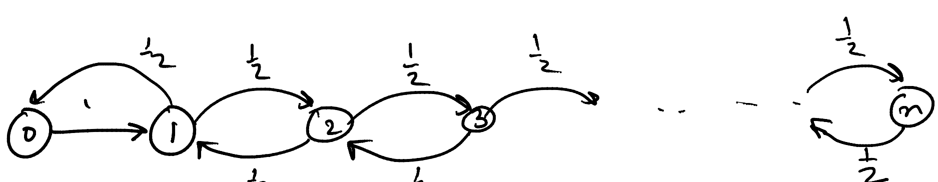
$$\Pr[X_{t+1} = i-1 \mid X_t = i] \leq \frac{1}{2} \qquad \forall i > 0.$$

Not necessarily MC but consider $Y_1, \ldots, Y_t$:

$$\Pr[Y_{t+1} = 1 \mid Y_t = 0] = 1$$

$$\Pr[Y_{t+1} = i+1 \mid Y_t = i] = \frac{1}{2} \quad \forall i > 0$$

$$\Pr[Y_{t+1} = i-1 \mid Y_t = i] = \frac{1}{2} \quad \forall i > 0.$$




Let $h_i = \mathbb{E}[\#$ of steps to reach $n$ starting from $i]$

$$h_0 = h_1 + 1$$

$$h_i = \frac{1}{2}(h_{i-1} + 1) + \frac{1}{2}(h_{i+1} + 1)$$

$$= 1 + \frac{1}{2}(h_{i-1} + h_{i+1})$$

Check that $h_i = n^2 - i^2$ satisfies recurrence.

So, $h_0 = n^2 - 1 + 1 = n^2$.

$\Pr[\text{Time to reach } n \text{ more than } 2n^2] < \frac{1}{2}$

$\Pr[\text{Not reach } n \text{ in more than } 2n^2 c \text{ steps}] < \frac{1}{2^c}$.

Q: Why doesn't this work for 3SAT?

<u>Classification of states</u>

Say $i \mapsto j$ if there is a path from state $i$ to state $j$

<u>Def</u>: State $i$ is called <u>transient</u> if $\exists$ state $j$ such that $i \mapsto j$ but $j \not\mapsto i$. Otherwise, called <u>recurrent</u>.

Note that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$. So, can partition states into $R_1 \cup R_2 \cup \cdots \cup R_\ell \cup T$ where $R_i$'s are recurrent and $T$ is transient.

<u>Obs</u>: If $i \in R_1$, $j \notin R_1$, then $P_{ij} = 0$.

<u>Def</u>: $r_{ij}^t = \Pr[X_t = j \text{ and } X_s \neq j \ \forall\, 1 \le s < t \mid X_0 = i]$

$$h_{ij} = \sum_{t \ge 1} t \cdot r_{ij}^t \qquad \text{(hitting time)}$$

<u>Obs</u>: $i$ is transient iff $\sum_{t \ge 1} r_{i,i}^t < 1$.

<u>Lemma</u>: In a finite MC, if $i$ is recurrent, then $h_{ii} < \infty$.

Def: MC is irreducible if it consists of only one

Def : MC is ... recurrent class. (Graph is strongly connected)

<u>Stationary probability</u>

Def : $\pi$ is called stationary if $\pi P = P$.

Thm : Any finite MC has at least one stationary distribution.

Pf : Use LP duality

$$\max \sum \pi_i$$
$$\text{s.t. } \pi P - P = 0$$
$$\pi \geq 0$$

Primal unbounded $\iff$ Dual infeasible

$$\min \ 0$$
$$\text{s.t. } (P - I) y \geq 1$$

Suppose $\exists$ dual solution $y = [y_1, \dots, y_k]$. Suppose $y_1$ largest. Then $(Py)_1 \leq y_1 < 1 + y_1 \rightarrow\leftarrow$

Def : If $\forall i$, $\gcd\{t : P^t_{i,i} > 0\} = 1$, then MC <u>aperiodic</u>.

Def : If $\exists t_0$ s.t. $\forall i,j$, $\forall t > t_0$, $P^t_{i,j} > 0$, then MC <u>ergodic</u>

Thm : MC ergodic iff irreducible and aperiodic.

Thm : Finite, ergodic MC

1. Chain has unique stationary dist. $\pi$
2. $\forall j$ and $i$, $\lim_{t \to \infty} P^t_{j,i}$ exists, indep of $j$
3. $\pi_i = \lim_{t \to \infty} P^t_{j,i} = \frac{1}{h_{i,i}}$