Lecture 16

Friday, March 04, 2016
8:42 AM

Agenda : (1) Finish hitting time, cover time
(2) Mixing times in MC

Note : Problem Set 7 out

- Last time, we defined hitting time h_{uv} and commute time C_{uv} for random walks on graphs

- $h_{uu} = \frac{2m}{\deg(u)}$

- For an edge (u,v), $h_{uv} \leq 2m$ [adopt flawed proof from last class of looking at MC on edges], so $C_{u,v} \leq 2m + 2m = 4m$

- Also, saw $C_{uv} = 2m\, R_{u,v}$ ← eff resistance between u and v
for any pair (u,v).

- Define $C(G,s) = \mathbb{E}$[time to visit all vertices starting from s],
$C(G) = \max_s C(G,s)$

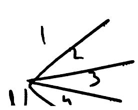
- Thm : For any connected, non-bip G, $C(G) \leq 4m(n-1)$
$= O(n^3)$

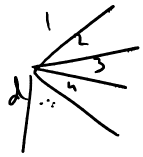
- App : $O(\log n)$-space algo for s-t connectivity

- Pf : Look at DFS tree T rooted at any s, giving edges
$(u_0, u_1), \ldots, (u_{2n-3}, u_{2(n-1)})$
Each edge occurs twice in opposite directions

$$C(G,s) \leq \sum_{i=0}^{2n-3} h_{u_i, u_{i+1}} \leq 4m(n-1)$$

- App : Universal traversal sequences
Labeled d-regular graphs. Want to visit all vertices by following labeled edges in a fixed sequence





vertices by following edges in a fixed sequence

<u>Thm</u>: $\exists$ UTS that works for all d-regular graphs of size poly(n).

<u>Pf</u>: Take a random sequence of length $8n^6$, breaking it into n^3 chunks of length $8n^3$ each. For any fixed graph, and fixed v, prob v is not visited in any chunk is $\leq \frac{1}{2^{n^3}}$. Take union bound over all n vertices v and all $2^{O(n^2)}$ graphs G.

<u>Rem</u>: Very loose. Can be improved to $O(n^3 d \log n)$.

- Theorem is tight for lollipop graph and line but very bad for complete graph.
- $R(G) = \max_{u,v} R_{u,v}$
- <u>Thm</u>: $mR \leq C(G) \leq 2e^4 R m \ln(n) + O(1)$
- <u>Pf</u>: Lower bound trivial because $\max(h_{uv}, h_{vu}) \geq mR_{uv}$

<u>UB</u>: Break into $\ln(n)$ chunks of length $2e^4 Rm$.

In each chunk, for fixed v,

$$\Pr[v \text{ not visited}] \leq \frac{1}{e^4}$$

$$\Pr[v \text{ not visited in any chunk}] \leq \frac{1}{n^4}$$

$$\Pr[\exists v \text{ not visited in any chunk}] \leq \frac{1}{n^3}$$

So, $\mathbb{E}[\text{cover time}] \leq 2e^4 mR \ln n + O(1)$

Other notes:

- has solution to system of lin eq's of poly size
- $C(G)$ can be written as hitting time for exp size MC.
- Will show in pset that $C(G) \leq O(h_{max} \log n)$
- On the other hand, trivially, $C(G) \geq h_{max}$. So, main question is whether $O(\log n)$ necessary for particular graphs.
- <u>Matthews bound</u>: $C(G) \geq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \cdot h_{min}$

<u>Pf</u>: Consider vertices in random order $\pi(1), \pi(2), \ldots, \pi(n)$.

$$\Pr[\pi(i) \text{ visited after } \pi(1), \ldots, \pi(i-1)] = \frac{1}{i}$$

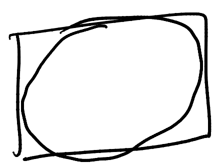
$$\mathop{\mathbb{E}}_{\pi}[C(G)] = \sum_{i=1}^{n} \mathbb{E}\left[\text{steps between } \pi(i-1) \text{ and } \pi(i) \,\middle|\, \pi(i) \text{ visited after } \pi(i-1)\right]$$

$$\geq H_{n-1} \cdot h_{min}$$

<u>Mixing Time</u>

- Now shift to a different application of MC's: Markov Chain Monte Carlo
- <u>Goal</u>: Sample uniformly from some space $A \subseteq U$.
- <u>Example</u>: Generate a random 3-coloring of a graph or generate a random spanning tree of a graph
- <u>Idea</u>: Choose an arbitrary $s \in U$ and move in a Markov chain whose stationary distribution is uniform
- Why would we want to sample?
  - For simulation purposes, finding out characteristics of random items

- Example : Sample a random vertex of a very large graph and you know only its local structure
  - MC: At x, take an edge (x, y) w. prob $\min\left(\frac{1}{d(x)}, \frac{1}{d(y)}\right)$. O.w. stay at x.
  - Stationary distribution is uniform.
- Example : Optimization. Choose edge w. prob. $\lambda^{-(f(x)-f(y))}$ if $f(y) < f(x)$
- Example : Counting
- Suppose we want to count solutions to a problem.
  Example: Given 2SAT formula φ, how many solutions are there? Given a graph, how many 3-colorings are there?
- General problem : In a universe U, how many belong to a particular subset A, where for a given $x \in U$, you can quickly decide if $x \in A$?
  - Sampling : Monte Carlo method



Estimating π

Generally, if we want to get $\bar{m}$ s.t.
$(1-\varepsilon)\cdot|A| \leq \bar{m} \leq (1+\varepsilon)\cdot|A|$ w.p. $\geq 1-\delta$, need

$$\frac{3}{\varepsilon^2}\cdot\frac{|U|}{|A|}\cdot\ln\left(2/\delta\right) \text{ samples.}$$

Infeasible if $|A|$ is exponentially smaller than $|U|$.

- If problem has some structure, then can use approximate sampling to approximately count.
- Def : FPRAS to $f(x)$ outputs $\hat{f}(x)$ s.t. $|\hat{f}(x) - f(x)| \leq \varepsilon f(x)$ w. prob. $1-\delta$ in time poly$(|x|, 1/\varepsilon, \ln 1/\delta)$.

- **Def**: FPAUS generates sample ω of $\Omega(x)$ s.t.

$$\forall S, \quad \left| \Pr[\omega \in S] - \frac{|S|}{|\Omega(x)|} \right| \le \varepsilon.$$

in time poly$(|x|, \ln 1/\varepsilon)$.

- **Thm**: An FPAUS for independent set implies FPRAS for independent set.

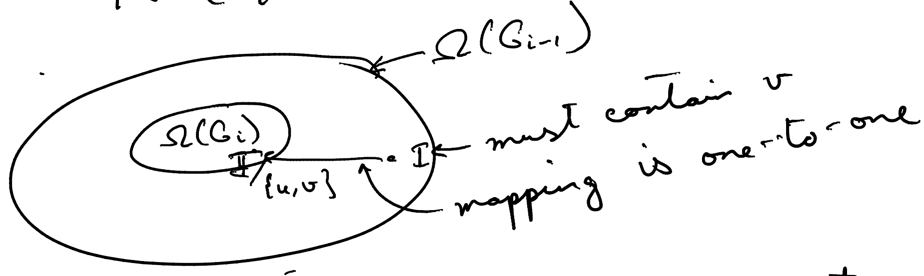
Pf: Let E_i be first i edges

$$G_i = G(V, E_i).$$

$$|\Omega(G)| = \frac{|\Omega(G_m)|}{|\Omega(G_{m-1})|} \times \frac{|\Omega(G_{m-1})|}{|\Omega(G_{m-2})|} \times \cdots \times \frac{|\Omega(G_1)|}{|\Omega(G_0)|} \times |\Omega(G_0)|$$

Claim: $r_i \stackrel{\text{def}}{=} \dfrac{|\Omega(G_i)|}{|\Omega(G_{i-1})|} \ge \dfrac{1}{2}$

Pf:



Now, want to approximate each r_i by Monte Carlo sampling. Estimate $\tilde{r}_i$ for each r_i.
Let $X_k = 1$ if k'th sample from $(\varepsilon/6m)$-uniform sampler of $\Omega(G_{i-1})$ is in $\Omega(G_i)$.

$$\left| \Pr[X_k = 1] - r_i \right| \le \frac{\varepsilon}{6m}$$

$$\tilde{r}_i = \frac{\sum X_k}{M}. \quad \mathbb{E}[\tilde{r}_i] \ge r_i - \frac{\varepsilon}{6m} \ge \frac{1}{3}$$

$$\Pr\left[|\tilde{r}_i - \mathbb{E}[r_i]| \ge \frac{\varepsilon}{12m} \cdot \mathbb{E}[\tilde{r}_i] \right] \le \frac{\delta}{m}$$

if $M = O\left(\frac{m^2}{\varepsilon^2} \ln \frac{m}{\delta}\right)$

With prob $\geq 1 - \frac{\delta}{m}$,

$$\tilde{r}_i \leq \left(1 + \frac{\varepsilon}{12m}\right)\left(r_i + \frac{\varepsilon}{6m}\right)$$

$$\leq \left(1 + \frac{\varepsilon}{12m}\right)\left(1 + \frac{\varepsilon}{3m}\right) r_i$$

$$\leq \left(1 + \frac{\varepsilon}{2m}\right) r_i$$

So, by u.b. with prob $\geq 1 - \delta$,

$$1 - \varepsilon \leq \prod \frac{\tilde{r}_i}{r_i} \leq \left(1 + \frac{\varepsilon}{2m}\right)^m \leq 1 + \varepsilon .$$

<u>Mixing ~~Times~~</u>

~~Total variation distance~~ : $\|\mu - \nu\|_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|$

<u>Def</u> : For a Markov chain, let

$$d(t) = \max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{TV}$$

$$\bar{d}(t) = \max_{x, y \in \Omega} \|P^t(x, \cdot) - P^t(y, \cdot)\|_{TV}$$

<u>Def</u> : $t_{mix}(\varepsilon) = \min\{t : d(t) \leq \varepsilon\}$

$t_{mix} = \min\{t : d(t) \leq 1/4\}$.