Lecture 10

Wednesday, April 26, 2017 12:26 PM

Feasibility Problem: Given a polytope $P \subseteq \mathbb{R}^n$,
either find a point $x \in P$
or prove $P = \emptyset$.

How is P given?
- It could be explicitly described.
- We will assume a weaker access model

Separation Oracle:
Given $x \in \mathbb{R}^n$, the sep. oracle either says YES $x \in P$, or says NO and desribes a constraint $(a \in \mathbb{R}^n, b \in \mathbb{R})$ st

$$a^T x \leq b \quad \text{but} \quad a^T z \geq b, \forall z \in P$$

Note: if P is described as having m explicit constraints, then the separation oracle can be simulated by checking the m-different constraints.

Assumptions

① $P \subseteq B(0,R)$, where $R \approx$ "polysized"
often satisfied because our vars will be in $[0,1]$

② P is full dimensional (if nonempty)
This means that it has an interior pt x and a radius $r > 0$ s.t $B(x,r)$ ... the ball of radius r around x ... is fully inside P
This is often not satisfied. But this is also an assumption which isn't needed but makes impl & the analysis hairier.

Geometric Facts

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① If P is full dimensional and non-empty, then

$$\text{vol}(P) \geq \rho^{-\text{poly}(n)}$$

where ρ is $\max\limits_{i,j}\{A_{ij}, b_i\}$ where $P \equiv \{x : Ax \geq b\}$

Pf :- • If P is full-dimensional, then it contains an $(n+1)$-vertex simplex $\{v_0, v_1, \dots, v_n\}$ with each v_i being a bfs of P

• $\text{vol}(P) \geq \text{vol}(\text{Simpl}(v_0, v_1, \dots, v_n))$

$$= \frac{1}{n!}\left|\det\underbrace{\left(v_0 - v_1, \; v_0 - v_2, \dots, v_0 - v_n\right)}_{M}\right|$$

• Each entry of this matrix M is a rational number if each entry of A, b are rational.

$\because$ Each $v_i = B^{-1} b_B$ for some $n \times n$ non-sing. matrix B

Furthermore the rational #s of the form p/q satisfy $|p|, |q| \leq \gamma \leq \rho^{\text{poly}(n)}$

Lin. Alg. Fact :- If M is a non-singular matrix of rational entries p/q with $|p|, |q| \leq \gamma$, then

$$|\det(M)| \geq \gamma^{-\text{poly}(n)}$$

Pf (Sketch) : – Convert M to row-echelon form $\tilde{M}$
– Each entry of $\tilde{M}$ is a rational $\tilde{p}/\tilde{q}$ with $|\tilde{p}|, |\tilde{q}| \leq \gamma^{-n^2}$

– $\therefore |\det(M)| \geq \gamma^{-n^3}$

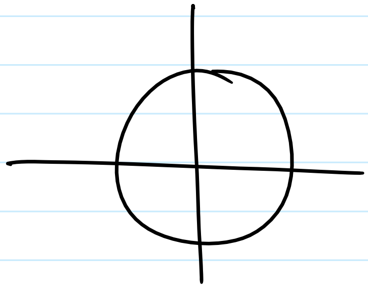
$\tilde{p}/\tilde{q}$ with $|\tilde{p}|, |\tilde{q}| \le \gamma$

$- \therefore |\det(M)| \geq \gamma^{-n^3}$

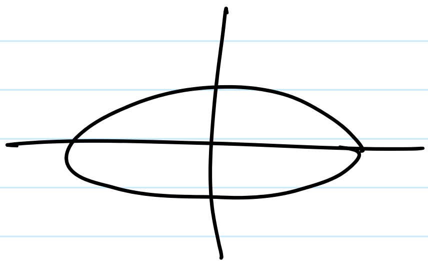
∎

② Ellipsoids: (Sheared & rotated Balls)

2D!



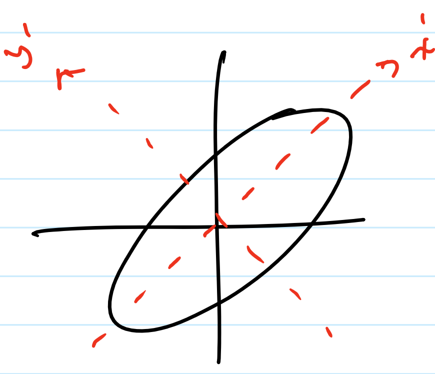
ball : $x^2 + y^2 \le 1$



axis-parallel ellipse :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$

$$(x, y) \underbrace{\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}}_{D} \begin{pmatrix} x \\ y \end{pmatrix} \le 1$$



not-axis-parallel ellipse

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} \le 1$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} \qquad R \equiv \text{rotation matrix}$$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ R = rotation matrix

$$(x, y) \underbrace{R^T D R}_{\text{III}} \begin{pmatrix} x \\ y \end{pmatrix} \leq 1 \qquad ? \; R^T R = I$$

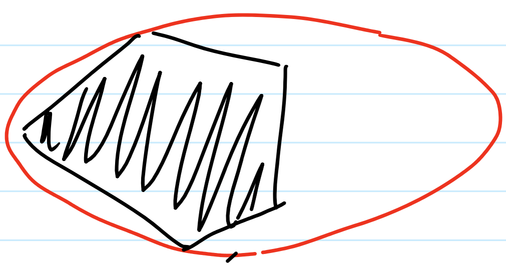
positive matrix or positive definite matrix.

An ellipsoid $\mathcal{E}$ in n-dimensions is characterized by a positive definite matrix A and a ctr c

- $\mathcal{E} := \{x \in \mathbb{R}^n : (x-c)^T A^{-1} (x-c) \leq 1\}$
- $\text{vol}(\mathcal{E}) = \det(A)$

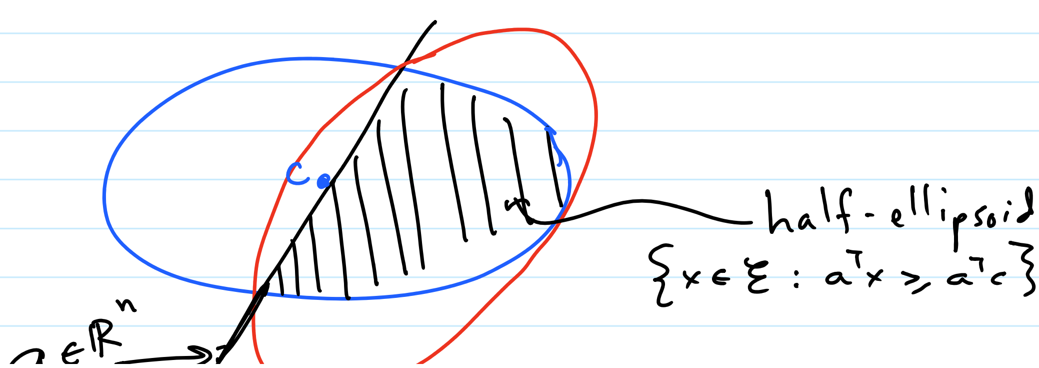
③ Enclosing Ellipsoids

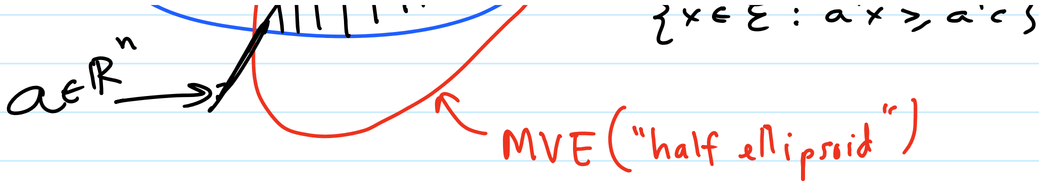
Given a convex body K, the Minimum Vol. Enclosing Ellipsoid assoc. with K is a well studied geometric object.



(pardon my terrible ellipsoids)

For our purposes, the body K will be a nice object – it'll be a "half ellipsoid".





Given ellipsoid $\mathcal{E} \equiv \mathcal{E}(A, c)$, and a cutting hyperplane $a \in \mathbb{R}^n$, the MVE of the $\mathcal{E} \cap \{x : a^T x \geq a^T c\}$ has a closed-form-formula.

positive matrix — center

(which I could write and derive, but I won't in the gen. case)

Ellipsoid Algorithm

① $\mathcal{E}_0 \equiv B(0, R)$; $x_0 = \vec{0}$

② While STOP:

- Ask Sep. oracle is $x_i \in P$?
- If YES, return x_i; STOP
- If NO:
  - Get a s.t. $a^T z \geq a^T x_i \quad \forall z \in P$
  - $\mathcal{E}_{i+1} \equiv \text{MVE}\left(\mathcal{E}_i \cap \{x : a^T x \geq a^T x_i\}\right)$
  - If $\text{vol}(\mathcal{E}_{i+1}) < \gamma^{-\text{poly}(n)}$, return $P = \emptyset$; STOP

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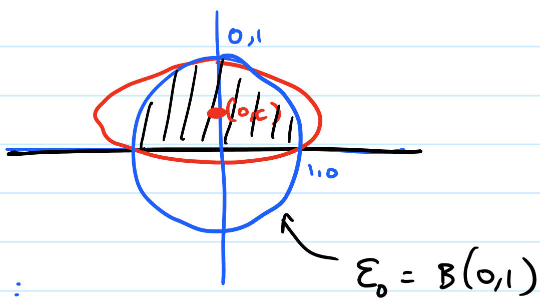
Main Lemma :- $\mathrm{vol}(\mathcal{E}_{i+1}) \leq \left(1 - \frac{1}{2n}\right)\mathrm{vol}(\mathcal{E}_i)$

"**Proof**" :- Only for $n = 2$, $\mathcal{E}_i = B(0,1)$ & $a = (0,1)$

› I don't know of a good intuition for this fact. Let's just do a simple calculation for $n=2$ ... just to get an idea.

By symm, the center of the red ellipse is at $(0,c)$ for some c.



$\therefore$ Eqn of the ellipse is :

$$\frac{x^2}{a^2} + \frac{(y-c)^2}{b^2} = 1$$

— This should pass through $(0,1)$, $(-1,0)$, & $(1,0)$

$$\Rightarrow \quad \frac{(1-c)^2}{b^2} = 1 \quad \Rightarrow \quad b = 1 - c$$

&

$$\frac{1}{a^2} + \frac{c^2}{b^2} = 1 \Rightarrow \frac{1}{a^2} = 1 - \frac{c^2}{(1-c)^2} = \frac{1-2c}{(1-c)^2}$$

$$\Rightarrow \quad a = \frac{1-c}{}$$

$\sqrt{1-2c}$

- Area of the ellipse is so,

$$\pi ab = \pi \cdot \frac{(1-c)^2}{\sqrt{1-2c}}$$

- We choose c, and one can now do calculus to find the best c.

But already @ $c = 1/3$, say, the ellipse area $\leq \pi \cdot \frac{4\sqrt{3}}{9} = \pi \cdot \sqrt{\frac{48}{81}}$

$$< \frac{3\pi}{4}$$

Solving Linear Programs with maybe exponentially many constraints

(*) $\{\min c^T x : Ax \geq b\}$, but A has exp many constr. Can still be solved in poly time

IF: $\exists$ an efficient algorithm which given x can either prove $Ax \geq b$ or find a_i with $a_i^T x < b_i$.

Design Problems

Max-Min-Spanning Tree problem

Input: • $G = (V, E)$ unweighted, undirec. Graph

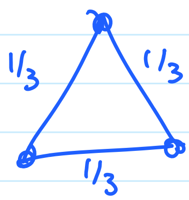
• Budget B

Output: $w : E \to \mathbb{R}_{\geq 0}$ s.t.

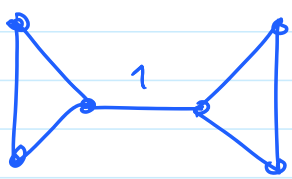
$$\sum_e w(e) \leq B$$

Obj:- $\displaystyle\max_{w} \min_{T:\text{spanning tree}} w(T)$

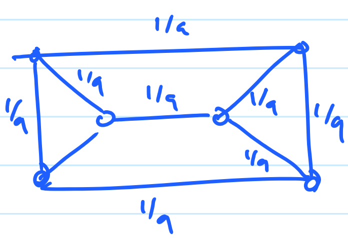
Example:- $(B=1)$



opt $= 2/3$



opt $= 1$



opt $= 5/9$ (?)

Mathematical Formulation

$$\text{Max } \lambda$$

$$\forall T \in G : \sum_{e \in T} w_e \geq \lambda \qquad (*)$$

$$\sum w_e \leq B \qquad (**)$$

$$e \in E$$

$$\underbrace{1 \geq}_{\text{maybe}} w_e \geq 0 \qquad (***)$$

<u>Observe</u> :- Although this LP has exp. many constraints, Ellipsoid allows me to solve it since Minimum Spanning Tree is solvable in polynomial time !

"Design is a Easy as Optimization"