Lecture 5

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Linear Programming Relaxations

A math. prog. formulation for Set Cover.

$$IP := \text{Min} \sum_{i=1}^{m} c(S_i)\, x_i$$

indicator variable whether set S_i is picked in the cover

$$\forall j = 1..n, \quad \sum_{i: e_j \in S_i} x_i \geq 1$$

Each elt. needs to be covered.

$$\boxed{x_i \in \{0,1\}}$$

hard constraint.

Claim: IP = OPT

- Since Set-Cover is NP-hard, solving IP is NP-hard too.

- LP-relaxation removes the hard constraints.

$$LP := \min \sum c(S_i) \cdot x_i$$

$$\forall j = 1..n, \quad \sum_{i: e_j \in S_i} x_i \geq 1$$

$$1 \geq x_i \geq 0$$

LP-relaxation for set-cover.

Observe :- LP $\leq$ OPT , for any instance.

$\therefore$ LP-relaxations give an "automatic" way of obtaining "lower bounds on OPT

"Amazing Theorem: Linear Programs can be solved in polynomial time"

- Most of the approximation algorithms we will see ahead in the course will use these relaxations to obtain "real-valued" solns which will bound the value of OPT.
  The CREATIVITY lies in taking this real-valued solution & rounding them to $\{0,1\}$-valued solutions.

Vertex Cover Problem

Input: $\cdot$ $G = (V, E)$
$\cdot$ costs c_v on each vertex.

Output: $S \subseteq V$ s.t. each edge has at least one endpoint in S

<u>Obj</u>: Minimize $c(S)$

<u>IP</u>: $\min \sum c_v x_v \;:\; x_v \in \{0,1\}$

$\forall e=(u,v):\; x_u + x_v \geq 1$

<u>LP</u>: $x_v \in \{0,1\} \longleftrightarrow x_v \in [0,1]$

<u>Algorithm</u>: $S \equiv \{v \mid x_v \geq 1/2\}$

<u>C1</u>: $c(S) \leq 2 \cdot LP \leq 2\, OPT$ ... easy

<u>C2</u>: S is a vertex cover ... also easy.

<u>Thm</u>:- There is 2-appx algorithm for Vertex Cover.

Integrality Gaps

For any (minimization) problem, Π, and any LP-relaxation for it, the integrality gap of the LP-relxn is defn. as

$$IG := \sup_{I \text{ of } \Pi} \frac{OPT(I)}{LP(I)}$$

Often LP-based algorithms allow us to prove upper bounds on integrality gaps.

For instance, the theorem above shows that the IG of the natural LP is ≤ 2

$$\text{since } \forall \mathcal{I} \quad \underset{\text{OPT}(\mathcal{I})}{\leq} \text{ALG}(\mathcal{I}) \leq 2 \cdot \text{LP}(\mathcal{I})$$

Lower Bound: For the above LP, however, the IG $\to 2$.

$G = K_n$, $\quad x_v = 1/2 \ \forall v$. $\qquad \text{LP} = n/2$

$\text{OPT} = n-1$

Strengthening LP-relaxations

Once we meet an LP-relaxation, and "juiced" it all out, ie, proved upper & lower bounds on it, we should try to strengthen the LP.

This is done by adding valid constraints

Linear inequalities that are satisfied by all integer solns.

In vc, eg, we can add the Δ-constraints

$$\forall u,v,w \text{ st } (u,v),(v,w),(w,u) \in E: \qquad x_u + x_v + x_w \geq 2$$

Note that LP' which has these ineq is stronger. ie. $LP_0 \leq LP' \leq OPT$.

$\therefore \quad IG(LP') \leq IG(LP_0) \leq 2$
