Lecture 9

Wednesday, April 19, 2017 3:18 PM

Minimum Spanning Tree Polytope

In this lecture we will look at another polytope which is exact.

Input:- $G = (V, E)$, $|E| = m$.

Goal:- Describe a polytope (linear system of inequalities) whose vertices correspond to spanning trees of G

$$P = \{ x \in [0,1]^m :$$

$$- \quad x(E) = n - 1$$

$$- \quad x(\underbrace{E[S]}_{\text{edges with both endpoints in } S}) \le |S| - 1 \qquad \forall S \subseteq V$$

$$\}$$

- It is clear that the "indicator vector" $\chi_T \subseteq \{0,1\}^m$ for any tree T defined as:

$$\chi_T(e) = \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{o/w} \end{cases}$$

is feasible in P.

It is not too difficult (convince yourself) that any $\{0,1\}$-vector in P corresponds to a spanning tree.

- **Thm** :- The MST polytope is integral, that is, every bfs is a $\{0,1\}$-vector.

Proof : Let x be a bfs.

$F = \{e \mid 0 < x_e < 1\}$. Suppose $F \neq \emptyset$

For this lecture, i am going to prove $F \neq E$. and leave the rest as an exercise. So, henceforth, suppose for the sake of contradiction, $F = E$

- Observe :- $x(F) = n-1$ & $x_e < 1 \ \forall e \in F$

$$\Downarrow$$

$$\boxed{|F| \geq n} \quad \ldots\ldots \ (*)$$

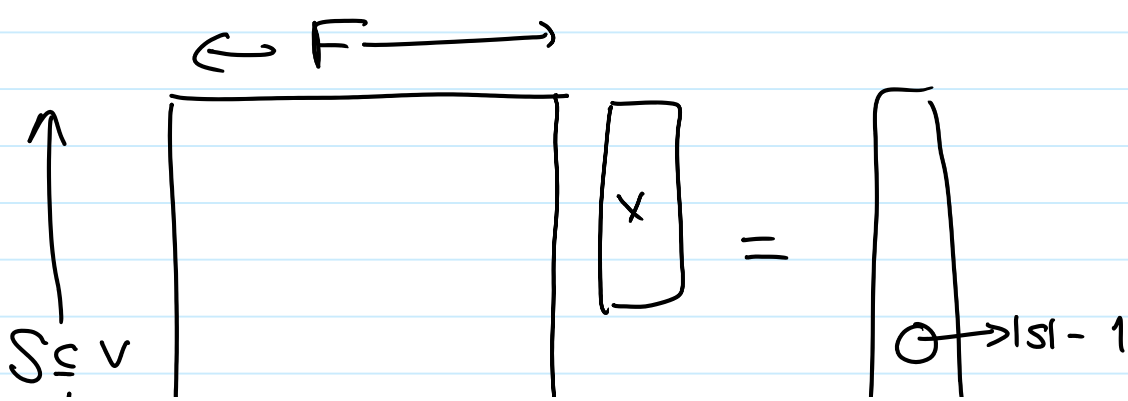
- $|F| = \text{rank}(B_F)$

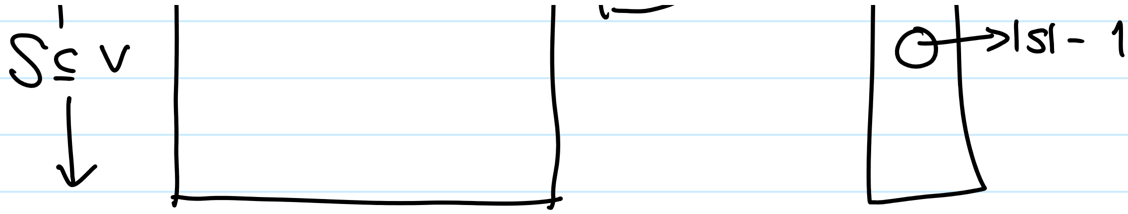
where B_F is the matrix where
- columns are edges of F
- rows are the tight sets of x

ie, non-triv. inequalities of P sat. with eq. by x

- We now prove $\text{rank}(B_F) \leq n-1$ contradicting $(*)$ and thus proving $F \neq E$.

- How does B_F look like?



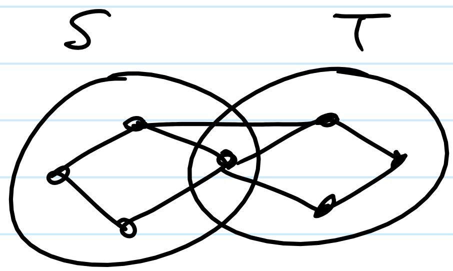


- Let's take two "tight subsets" S & T

$$x(E(S)) = |S| - 1$$
$$x(E(T)) = |T| - 1$$

& Suppose S, T non-trivially intersect ie

$$S \setminus T \neq \emptyset \quad \& \quad T \setminus S \neq \emptyset$$



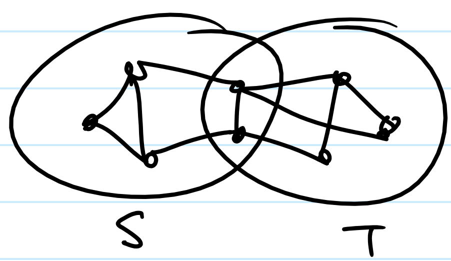
- Observe:

ⓐ $|S| + |T| = |S \cap T| + |S \cup T|$

(De Morgan's law)

ⓑ $E(S) \cup E(T) \subseteq E(S \cup T)$

For any $(u,v) \in E(S)$,
both $u,v \in S$
$\therefore$ both $u,v \in S \cup T$
$\Rightarrow (u,v) \in E(S \cup T)$



Example 1 shows it can be strict subset.

ⓒ $ES \cap ET = E(S \cap T)$

if $(u,v) \in E(S) \cap E(T) \Rightarrow u,v$ are both in S and T
ie, $\{u,v\} \subseteq S \cap T \Rightarrow (u,v) \in E(S \cap T)$

on the other hand, if $(u,v) \in E(S \cap T)$,
then $\{u,v\} \subseteq S \cap T$

$\Rightarrow \{u,v\} \subseteq S$ and $\{u,v\} \subseteq T$

$\Downarrow \qquad\qquad \Downarrow$

$(u,v) \in E(S) \qquad (u,v) \in E(T)$

(b) & (c) $\Rightarrow$

$$x(E(S)) + x(E(T)) \leq x(E(S \cup T)) + x(E(S \cap T))$$

$\because$ LHS $= x(E_S \cup E_T) + x(E_S \cap E_T)$ by de Morgan

$\underset{\{\text{since } x \geq 0\}}{\leq} x(E(S \cup T)) + x(E(S \cap T))$

$\therefore$ if S, T are tight, and (a), gives

$$x(E(S \cup T)) + x(E(S \cap T)) \geq |S \cup T| - 1 + |S \cap T| - 1$$

But $S \cup T$, $S \cap T$ are also valid ineq.

$\Longrightarrow$ $\boxed{\text{if } S, T \text{ are tight \& non-trivially intersect, then } S \cup T \text{ \& } S \cap T \text{ are tight.}}$

Furthermore, since $x_e > 0$ for all e,

we have $E(S) \cup E(T) = E(S \cup T)$ as well.

$$\Longrightarrow \quad \text{row}(S) + \text{row}(T) = \text{row}(S \cap T) + \text{row}(S \cup T)$$

$\therefore$ In any basis of B_F, if two sets

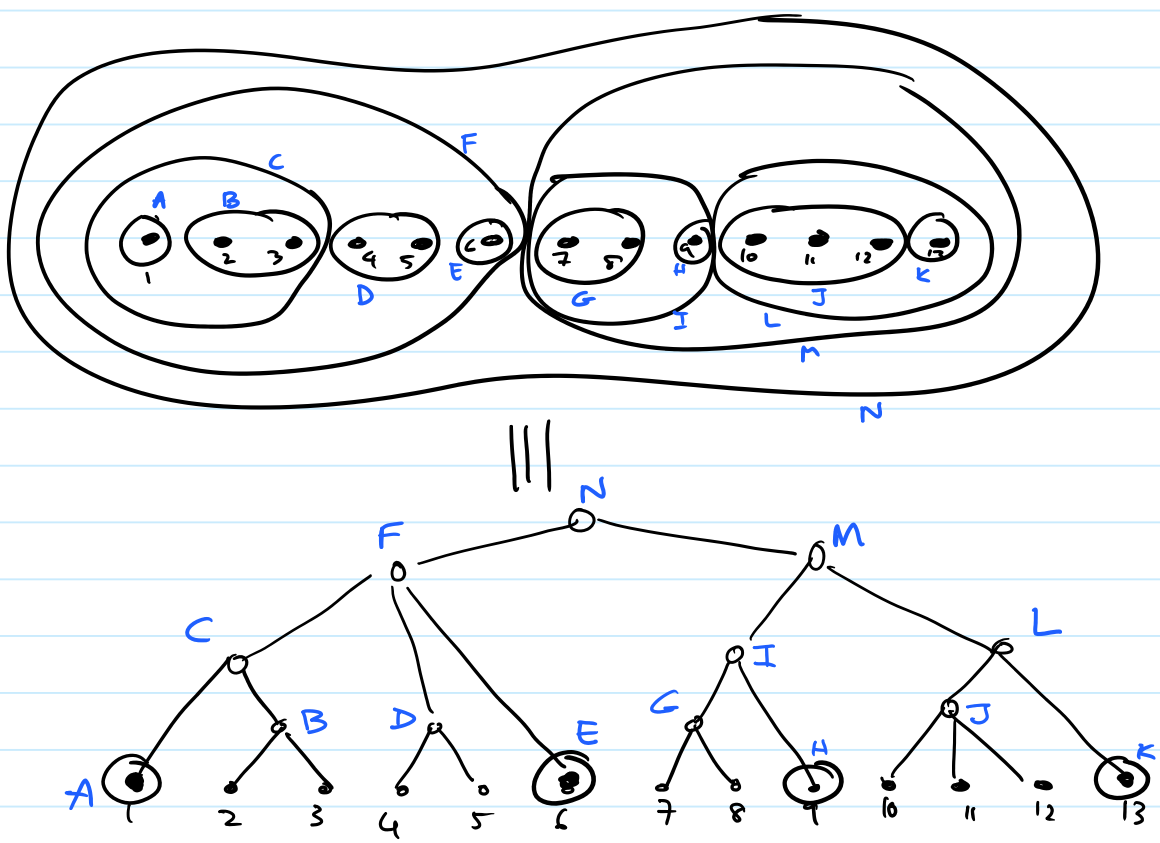
S, T non-trivially intersect, we can throw one away & replace with $S \cup T$ & $S \cap T$.

$\therefore$ We may assume there exists a basis B_F s.t. all the linearly independent rows corresponds to sets which <u>don't nontrivially intersect</u>.

<u>Defn</u>: (Laminar Family)

$\mathcal{L}$, a coll^n of subsets of $[n]$, is a <u>laminar family</u> if no $S, T \in \mathcal{L}$ non-triv. intersect.

Laminar Families as Trees



We have proved above ..

Lemma :-

Given any x, a bfs of P_{MST}, we may assume that the sets corr. to the lin. ind. set of "tight rows" form a laminar set. Furthermore, each <u>minimal</u> set S has $|S| \geq 2$ ... o/w the corr. ineq is $0 \equiv 0$.

Claim :- If $\mathcal{L}$ is a laminar set over n elts and the minimal sets are of size ≥ 2, then

$$|\mathcal{L}| \leq n-1.$$

Pf :- Induction. (1)

(2) # of int-nodes in a tree where every non-root, non-leaf has deg ≥ 3 is $\leq |\text{Leaves}| - 1$ ... which is again proved by ind.

$$\left.\begin{array}{l} \therefore \ \text{rank}(B_F) \leq n-1 \\ \Rightarrow |F| \leq n-1 \end{array}\right\} \text{contradicting } |F| \geq n.$$

∎

Key: (1) $x(E(S \cup T)) + x(E(S \cap T)) \geq x(ES) + x(ET)$

Defining $g(S) := x(ES)$, we have

$$g(S \cup T) + g(S \cap T) \geq g(S) + g(T)$$

g is SUPERMODULAR

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(2) $\quad x(E\,S) \leq \underbrace{|S| - 1}$

MODULAR function of S

i.e. $h(S) + h(T) = h(S \cup T) + h(S \cap T)$

A similar strategy / argument about extreme pt. Solns would also hold for :

$$x(\partial S) \geq 1 \quad , \quad \forall S$$

submodular

constant, and ∴ triv. modular.

HW Exercise

$$\{x \in [0,1]^E : \forall S \subseteq V,\ S \neq \phi \ \&\ S \neq V,\ x(\partial S) \geq 1 \}$$

Prove :- x be a basic feas soln. Then $\exists e$ st $x_e \geq \frac{1}{2}$.