CS31 (Algorithms), Spring 2020 : Lecture 15 Supp 1

Date:

Topic: Graph Algorithms 5 Supp 1: The Issue with Ford Fulkerson

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
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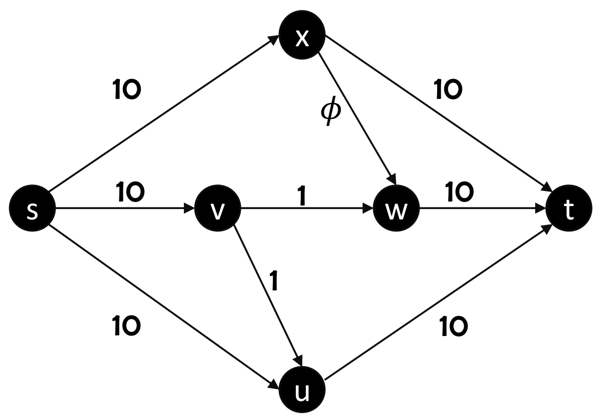
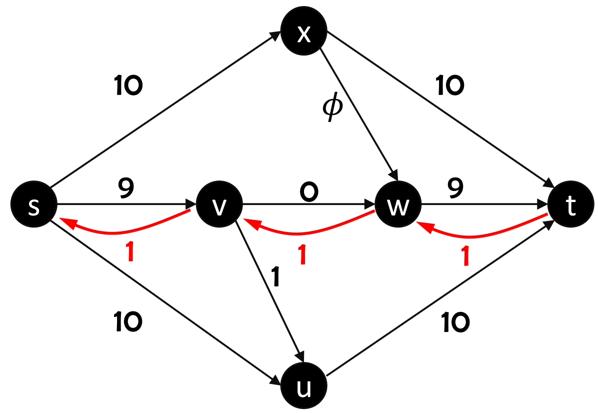
The FORDFULKERSON algorithm, we proved, returns a maximum flow when it terminates. We also proved that when $u(e)$'s are integer valued, it terminates in $O(nmU)$ time. However, there is one nub. The algorithm *may not even terminate* if the $u(e)$'s are irrational numbers. In this supplement, we describe such an example. In the Supplement 2, we see faster flow algorithms which always terminate. We didn't have time to cover them in the course.

As the example below shows, in each iteration the value of $\min_{e \in p} u(e)$ keeps decreasing geometrically leading to a situation which never ends. More frustratingly, the maximum flow value it converges to is also *far* from the maximum flow. The example is the following network in Figure . The value $\phi := \frac{\sqrt{5}-1}{2}$ is (one of) the irrational numbers satisfying $\phi^2 + \phi - 1 = 0$.

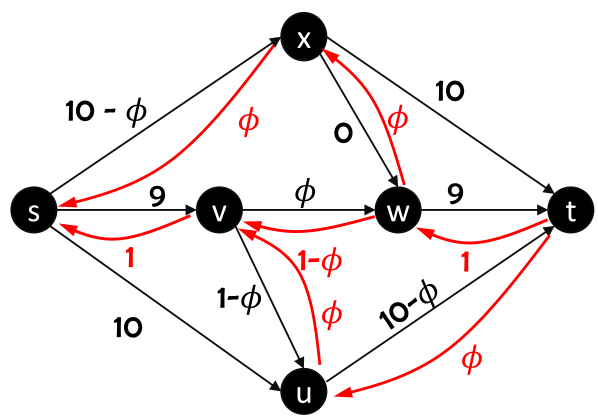


Before we describe the "bad paths" for FORDFULKERSON let's just note that the maximum flow value for the above network is 21. We can see this as follows: send 10 units of flow on the path (s, x, t); send 10 units of flow on the path (s, u, t), and then send 1 unit of flow along the path (s, v, w, t). To see this is the maximum flow, consider the cut $S = \{s, v, u\}$; note that $\partial^+ S := \{(s, x), (v, w), (u, t)\}$ of capacity 21. So, if we had indeed chosen the above three paths to augment on, FORDFULKERSON would've terminated to the correct answer in 3 rounds. Instead consider what happens next. We use F to maintain the value of the flow.

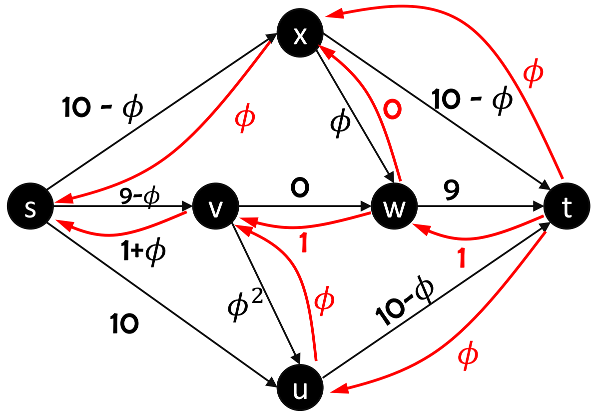
In Iteration 1, we provide the path (s, v, w, t) to the algorithm. We get $\delta_1 := 1$ and $F = 1$. After sending this flow, the residual network is this.



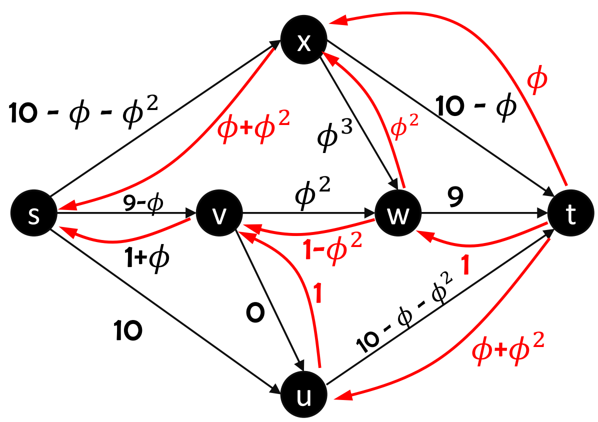
In Iteration 2, we provide the path (s, x, w, v, u, t). How much flow can we send on this path? Since $\phi < 1$, we see that $\delta_2 := \phi$. This makes the current value $F = 1 + \phi$. After sending this flow, the residual network is this (actually, draw it yourself and then check).



In Iteration 3, we provide the path (s, v, w, x, t). Note that $\delta_3 = \phi$ again, and the current flow becomes $F = 1 + 2\phi$. After sending this, the residual network is (you are drawing first, right, and then checking?)

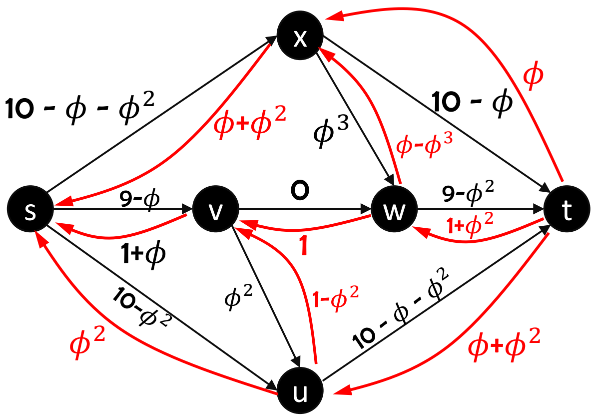


Now, in Iteration 4, we provide the path (s, x, w, v, u, t) again. The minimum $u_f(e)$ edge is (v, u) with residual capacity $\delta_4 \coloneqq 1 - \phi = \phi^2$. The value of $F = 1 + 2\phi + \phi^2$. After augmenting ϕ^2 flow on this path, we get the following residual network.



Note that the residual capacity of (x, w) is $\phi - \phi^2 = \phi(1 - \phi) = \phi^3$.

In Iteration 5, we provide the path path (s, u, v, w, t). Note on this path $\delta_5 = \phi^2$. After sending this flow the value becomes $F = 1 + 2\phi + 2\phi^2$, and the residual network becomes



Next, we repeat the paths sent in the previous 5 iterations again in this order. That is, we send ϕ^3 flow on (s,x,w,v,u,t). This makes the residual capacity of (v,u) equal to $\phi^2 - \phi^3 = \phi^4$. Follow up with a flow of ϕ^3 on (s,v,w,x,t) making the residual capacity of (x,w) equal to ϕ^3. The total flow is now $F = 1+2\phi+2\phi^2+2\phi^3$. Next send ϕ^4 flow on (s,x,w,v,u,t) following with a flow of ϕ^4 on (s,u,v,w,t). This will make the residual capacity of (x,w) equal to ϕ^5 and the residual capacity of (v,u) equal to ϕ^4, and the total flow $F = -1+2(1+\phi+\phi^2+\phi^3+\phi^4)$. And so and so forth – you get the drift.

Note that the value of F converges to (but never achieves) the value $F = -1+2/(1-\phi)$ which is nowhere close to the maximum flow value of 21.