1 Hall’s Theorem via Max-Flow-Min-Cut

We can also derive a theorem you may have seen in previous courses: Hall’s Theorem. A matching in a graph $G$ is perfect if all vertices participate as endpoints in the matching. This theorem states a necessary and sufficient condition for a bipartite graph $G$ to have a perfect matching. A definition: given any subset $S \subseteq L$, we define $\Gamma S := \{ r \in R : \exists \ell \in S, (\ell, r) \in E \}$ to be the set of neighbors of $S$.

**Theorem 1** (Hall’s Theorem). A bipartite graph $G = (L \cup R, E)$ with $|L| = |R|$ has a perfect matching if and only if for all $S \subseteq L$, $|\Gamma S| \geq |S|$.

**Proof.** We consider the network $\mathcal{N}$ defined above with one extra change: for all $e \in G$, we set $u(e) = \infty$. Note that the maximum flow doesn’t change since the total capacity incoming into any $\ell \in L$ is 1, and also the total capacity out going from any $r \in R$ is also 1. Thus, the infinite capacity in the “middle” doesn’t help in sending more flow. We see that it makes our arguments easier.

Now, we know that $G$ has a perfect matching if and only if the maximum $s, t$ flow in $\mathcal{N}$ is of value $|L|$. Using the max-flow-min-cut theorem, we get $G$ has a perfect matching iff the minimum $s, t$ cut in $\mathcal{N}$ is of value $|L|$.

Let us consider an $s, t$ cut in $\mathcal{N}$. Let $A$ be the subset inducing the cut; $A = s \cup S \cup T$ where $S \subseteq L$ and $T \subseteq R$. Note, $t \notin A$. The capacity of this cut is as follows.

$$u(\partial^+ A) = \begin{cases} \infty & \text{if } \Gamma S \notin T \\ (|L| - |S|) + |T| & \text{otherwise} \end{cases}$$

To see this, note that if $\Gamma S \notin T$, then there is an edge $(\ell, r) \in \partial^+ A$ of capacity $\infty$ (this is why we defined it so). If $\Gamma S \subseteq T$, then the only edges in $\partial^+ A$ are of the form $(s, \ell)$ for $\ell \notin S$ and $(r, t)$ for $r \in T$.

Now, if $A$ were the minimum $s, t$ cut, then observe that we would pick $T = \Gamma S$ (we would like to pick $T$ as minimal as possible). Therefore, the value of the minimum $s, t$ cut is precisely $\min_{S \subseteq L} (|L| - |S| + |\Gamma S|) = |L| + \min_{S \subseteq L} (|\Gamma S| - |S|)$.

See the figure below for an illustration.
Putting everything together, we get $G$ has a perfect matching if and only if

$$|L| + \min_{S \subseteq L} (|\Gamma S| - |S|) = |L|$$

or, in other words, for every $S \subseteq L$, we have $|\Gamma S| \geq |S|$.

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