CS 30: Discrete Math in CS (Winter 2019): Lecture 20

Date: 11th February, 2019 (Monday)

Topic: Combinatorics: Combinatorial Identities, Binomial Expansion

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
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In this lecture we will see a novel way of proving identities (equations). To prove two expressions are equal, we show so by describing a set such that the cardinality of the set can be argued to be both the LHS and the RHS. Sounds strange, right? But it is a very handy and powerful method. And such proofs are called "combinatorial" method of proving the identity.

1. **An easy warm up.**

   Theorem 1. For any number $n \in \mathbb{N}$ and natural number $k \leq n$, we have $\binom{n}{k} = \binom{n}{n-k}$.

   If we use the "formula" for $\binom{n}{k}$, then the above is obvious. Thus, the combinatorial method is perhaps not that interesting. But, as a first example it does quite okay.

   Proof. Consider the set of n-bit strings which have exactly k ones. Well, we have seen that the cardinality of this set is $\binom{n}{k}$. On the other hand, the same set is of size $\binom{n}{n-k}$ as well. Why? Well, there is a bijection between the set of n-bit strings with k ones and the set of n-bit strings with $(n-k)$ ones; given a string of the first type, you get a string of the second type by making all the ones zeros and all the zeros ones. □

2. **Pascals Identity.**

   Theorem 2. For any natural number $n \in \mathbb{N}$, and $k \in \mathbb{N}$ with $k \leq n$, we have

   $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

   Once again, it is not super hard to see the above identity algebraically by using the formula for $\binom{n}{k}$. However, the combinatorial proof just tells you "what's going on". And if you forger the above identity, you can use the combinatorial proof to remember it back (I do it frequently!).

   Proof. We know that $\binom{n}{k}$ is the number of subsets of an n-element set with exactly k elements. That is, if U is a finite set with $|U| = n$, and

   $$\mathcal{F} := \{S \subseteq U : \ |S| = k\}$$

   then $|\mathcal{F}| = \binom{n}{k}$.

Now, we count $|\mathcal{P}_k|$ differently. Let a be an arbitrary element in U. Consider the following two sets

$$\mathcal{F}_0 := \{S \subseteq U : \ |S| = k, \ a \in S\} \quad \text{and} \quad \mathcal{F}_1 := \{S \subseteq U : \ |S| = k, \ a \notin S\}$$

Note, by the sum principle

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \tag{1}$$

Now, note that $|\mathcal{F}_0| = \binom{n-1}{k-1}$. Why? To choose a subset of $S \subseteq U$ which contains a with $|S| = k$ is equivalent to choosing $T \subseteq U \setminus a$ with $|T| = k - 1$ and just adding a.

Finally, note that $|\mathcal{F}_1| = \binom{n-1}{k}$, since $\mathcal{F}_1 = \{S \subseteq U \setminus a : |S| = k\}$. That is, $\mathcal{F}_1$ are all subsets of $U \setminus a$ of size k. □

3. **The Binomial Theorem.**

Theorem 3. For any x, y and natural number n, we have

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

This is another thing you should tattoo in your brain. The needle you should be using is the proof below. You need to understand how one proves the identity above.

We consider x and y as variables. Then $(x + y)^n$ is a polynomial in these two variables which will be "opened up" using the distributive property.

$$(x + y)^n = (x + y) \cdot (x + y) \cdot (x + y) \cdots (x + y) \qquad n \text{ times}$$

- When we open it up as a polynomial, we will get a bunch of monomials. Let M be this set. Note that any monomial of the form $x^a y^b$ is M has property (i) integers $0 \le a \le n$, $0 \le b \le n$, and (ii) $a + b = n$. Convince yourself of this fact before moving on. This is because, each monomial is formed in the above product by picking some number of y's (call it a), and the rest $(n - a)$ of the x's. Thus,

$$M = \{x^a y^b : 0 \le a, b \le n, \ \ a + b = n\}$$

- The second thing to be figure out is the *coefficient* of $x^{n-a} y^a$. Convince yourself that the coefficient is the number of ways we can pick a string of length n with exactly $n - a$ x's and a y's. Why? Well, when we distribute the product of the n different $(x + y)$'s, we pick up either an x from each term, or a y. The monomial $x^{n-a} y^a$ emerges when we pick exactly $(n - a)$ x's and a y's. Note the *order* in which we pick these doesn't matter. Thus, the coefficient is the number of ways we can arrange $(n - a)$ x's and a y's in a line. And that is, precisely, $\binom{n}{a}$.

Proof over. We have found the monomials. We have found their coefficients. And we obtain the RHS of the binomial expansion.

4. **Consequences of the Binomial Expansion.**

There are many interesting corollaries of the Binomial expansion. In every statement below, n is a natural number.

Theorem 4.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Set $x = 1, y = 1$.

Theorem 5.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Set $x = 1, y = -1$.

Theorem 6.

$$\left(1 + \frac{1}{n}\right)^n > 2$$

Set $x = 1, y = 1/n$ and observe that the first two terms in the expansion is 1, while the rest are > 0.