1. Recap.

A graph $G = (V, E)$ is bipartite if there is partition of $V = L \cup R$ such that $L \cap R = \emptyset$ and for every edge $e = (u, v) \in E$, we have $|\{u, v\} \cap L| = |\{u, v\} \cap R| = 1$. That is, every edge has exactly one endpoint in $L$ and exactly one endpoint in $R$.

A matching $M$ in a graph is a subset of edges $M \subseteq E$ such that for any $e, e' \in M$, $e \cap e' = \emptyset$. That is, $M$ is a collection of edges which do not share end points. A vertex $v \in V$ participates in the matching $M$ if there is an edge in $M$ which is incident to $v$. In a bipartite graph $G = (L \cup R, E)$, a matching $M \subseteq E$ is an $L$-matching if all vertices in $L$ participate in $M$.

Given any subset $S \subseteq L$, we $N_G(S)$ are the set of vertices in $R$ which neighbors of some vertex in $S$. Hall’s Theorem says the following.

**Theorem 1.** Let $G = (V, E)$ be a bipartite graph with $V = L \cup R$. Then, $G$ has an $L$-matching if and only if

For every subset $S \subseteq L$, $|N_G(S)| \geq |S|$  

(Hall’s Condition)

**Proof.** Again, one direction is easy. That is, if $G = (L \cup R, E)$ has an $L$-matching, then we must have (Hall’s Condition). Why? Suppose there exists an $L$-matching called $M$. Then for any $S \subseteq L$, consider the set $T = \{v \in R : \exists u \in S : (u, v) \in M\}$. That is, look at all the partners in $M$, of vertices in $S$. Clearly, $T \subseteq N_G(S)$, and thus, $|N_G(S)| \geq |T|$. And $|T| = |S|$ since every vertex in $S$ has a partner in $M$ $(M$ is an $L$-matching). So, $|N_G(S)| \geq |S|$.

The interesting direction is the converse. Given that (Hall’s Condition) holds, we need to prove that $G = (L \cup R, E)$ has an $L$-matching. The proof is by induction on the number of vertices, but it has layers. So hold tight.

Let $P(n)$ be the predicate which is true if any bipartite graphs $G = (L \cup R, E)$ with $|L| = n$ satisfying (Hall’s Condition) has an $L$-matching.

We need to show $\forall n \in \mathbb{N} : P(n)$ is true; we proceed to prove this by induction.

**Base Case:** Is $P(1)$ true? Fix any graph $G = (L \cup R, E)$ with $|L| = 1$. Let $L = \{v\}$. (Hall’s Condition) implies, $\deg_G(v) \geq 1$. So, there is some edge $(v, w)$ incident on $v$. $M = \{(v, w)\}$ is an $L$-matching. So, $P(1)$ is true.

**Inductive Case:** Fix a natural number $k$. We assume $P(1), P(2), \ldots, P(k)$ are all true. We wish to prove $P(k + 1)$. To that end, we fix a bipartite graph $G = (L \cup R, E)$ which satisfies (Hall’s Condition) and $|L| = k + 1$.
Let \( u \in L \) be an arbitrary vertex. (Hall’s Condition) implies \( \deg(u) \geq 1 \), thus there is at least one edge \((u, v) \in E\). Pick one such edge arbitrarily. Consider the graph \( G' = G - \{u, v\} \).

That is, we delete \( u \) and then we delete \( v \). \( G' \) is also a bipartite graph, with \( G = (L' \cup R', E') \) where \( L' = L - u, R' = R - v \) and \( E' = E \backslash (N_G(u) \cup N_G(v)) \).

We now fork into two cases.

Case 1: \( G' \) satisfies (Hall’s Condition). This is the easy case. Since \( |L'| = |L| - 1 = k \), and since by the induction hypothesis, \( P(k) \) is true, we get that \( G' \) has an \( L' \)-matching; let’s call it \( M' \). Then, \( M := M' \cup \{u, v\} \) is the required \( L \)-matching in \( G \). So in this case, we have proven \( P(k + 1) \).

Case 2: \( G' \) doesn’t satisfy (Hall’s Condition). What does this mean? It means there is some subset \( S \subseteq L' \), such that \( |N_{G'}(S)| < |S| \). On the other hand, since \( G \) did satisfy (Hall’s Condition), we have \(|N_G(S)| \geq |S| \). Finally, note that the only way \( N_{G'}(S) \) and \( N_G(S) \) can be different is if \( N_G(S) \) has the vertex \( v \) in it. And in that case, \( N_{G'}(S) = N_G(S) \setminus v \).

Therefore, we have \( v \in N_G(S) \) and furthermore, \( |N_G(S)| = |S| \); if \( |N_G(S)| > |S| \), then indeed, \( |N_G(S)| \geq |S| + 1 \) because the LHS is an integer, which in turn implies \( |N_{G'}(S)| = |N_G(S)| - 1 \geq |S| \).

Now, we consider two different graphs. We consider \( G_1 = G[S \cup N_G(S)] \) and \( G_2 = G[(L \setminus S) \cup (R \setminus N_G(S))] \). Recall, the notion of induced subgraphs. It is also a good idea to draw a picture here for yourself.

Claim 1. Both \( G_1 \) and \( G_2 \) satisfy (Hall’s Condition).

Proof. Let’s first prove for \( G_1 \). Any subset \( T \subseteq S \) has \( N_G(T) \subseteq N_G(S) \). Thus, \( N_{G_1}(T) = N_G(T) \) as well. Since \( G \) satisfied (Hall’s Condition), we get \(|N_{G_1}(T)| = |N_G(T)| \geq |T| \). Thus, \( G_1 \) satisfies (Hall’s Condition).

Moving on to \( G_2 \). Fix a subset \( T \subseteq L \setminus S \). What is \( N_{G_2}(T) \)? Here is an useful observation:

\[ N_{G_2}(T) = N_G(T) \setminus N_G(S) = (N_G(S \cup T) \setminus N_G(S)) \]

The first equality follows since the neighbors of \( T \) in \( G_2 \) are precisely the neighbors of \( T \) in \( G \) which are not the neighbors of \( S \) in \( G \). The second equality is the clever part; it is noting that even if we look at neighbors of \( S \cup T \) in \( G \) and remove the neighbors of \( S \), we still get the neighbors of \( T \) in \( G \) which are not in \( N_G(S) \). Why is this useful? Because, \( N_G(S) \subseteq N_G(S \cup T) \). Thus, we know that \(|N_G(S \cup T) \setminus N_G(S)| = |N_G(S \cup T)| - |N_G(S)| \).

Putting all together, we get

\[ |N_{G_2}(T)| = |N_G(S \cup T)| - |N_G(S)| \geq |S \cup T| - |S| = |T| \]

where the inequality follows since \(|N_G(S \cup T)| \geq |S \cup T| \) by (Hall’s Condition) and since \(|N_G(S)| = |S| \), and the second equality follows since \( S \cap T = \emptyset \).

Since both \( G_1 \) and \( G_2 \) satisfy (Hall’s Condition), and since both \(|S| \) and \(|L \setminus S| \) are < \(|L| \), by the induction hypothesis, we get that \( G_1 \) has an \( S \)-matching called \( M_1 \) and \( G_2 \) has an \( L \setminus S \)-matching called \( M_2 \). Thus, \( M_1 \cup M_2 \) is the \( L \)-matching in \( G \). Done!