1 Functions

- **Definition.** A function is a mapping from one set to another. The first set is called the **domain** of the function, and the second set is called the **co-domain**. For every element in the domain, a function assigns a unique element in the co-domain.

Notationally, this is represented as

$$f : A \rightarrow B$$

where $A$ is the set indicating the domain, and $B$ is the set indicating the co-domain. For every $a \in A$, the function maps the value of $a \mapsto f(a)$ where $f(a) \in B$.

For example, suppose $A = \{1, 2, 3\}$, and $B = \{5, 6\}$, then the map $f(1) = 5$, $f(2) = 5$, $f(3) = 6$ is a valid function.

The **range** of the function is the subset of the co-domain which are actually mapped to. That is, $b \in B$ is in the range if and only if there is some element $a \in A$ such that $f(a) = b$.

More Examples.

- $f(x) = x^2$ is a function whose domain is $\mathbb{R}$, the set of real numbers, and so is the co-domain. The range, however, is the set of non-negative real numbers (sometimes denoted as $\mathbb{R}_+$).
- $f(x) = \sin x$ is a function whose domain is $\mathbb{R}$ and the range is the interval $[-1, 1]$.
- A (deterministic) computer program/algorithm is also a function; its domain is the set of possible inputs and its range is the set of possible outputs.

**Exercise:** Given a set $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, describe a function $f$ whose range is $\{5\}$, and describe a function $g$ whose range is $\{4, 6\}$. Just to get a feel, how many functions can you describe of the first form (whose range is $\{5\}$), and how many functions can you describe of the second form?

- **Sur-, In-, Bi- jective functions.** A function $f : A \rightarrow B$ is

  - **surjective**, if the range is the same as the co-domain. That is, for every element $b \in B$ there exists some $a \in A$ such that $f(a) = b$. Such functions are also called **onto** functions.

  For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, and consider the function $f : A \rightarrow B$ with $f(1) = 5$, $f(2) = 5$, and $f(3) = 6$. Then, $f$ is surjective. This is because for $5 \in B$ there is $1 \in A$ such that $f(1) = 5$ and for $6 \in B$ there is a $3 \in A$ such that $f(3) = 6$. 


Exercise: If $A$ and $B$ are finite sets, and $f : A \to B$ is a surjective function, can you show $|B| \leq |A|$?

injective, if there are no collisions. That is, for any two elements $a \neq a' \in A$, we have $f(a) \neq f(a')$. Such functions are also called one-to-one functions. For an injective function, one can define $f^{-1}(b)$ for all $b$ in the range of $f$.

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f : A \to B$ with $f(1) = 5$, $f(2) = 6$, and $f(3) = 8$. Then, $f$ is injective. This is because $f(1), f(2), f(3)$ are all distinct numbers.

Exercise: If $A$ and $B$ are finite sets, and $f : A \to B$ is an injective function, can you show $|A| \leq |B|$?

Injective functions have inverses. Formally, given any injective function $f : A \to B$, we can define a function $f^{-1} : B \to A \cup \{\bot\}$ as follows

$$f(b) = \begin{cases} a & \text{if } a \text{ is the unique } a \in A \text{ with } f(a) = b. \\ \bot & \text{otherwise, that is } f(a) \neq b \text{ for all } a \in A. \end{cases}$$

Sometimes people define a different inverse function where instead of adding the $\bot$ to the co-domain, they only consider the range of $f$ as the domain. That is, the following is a valid function $g : \text{range}(f) \to A$ where $g$ maps $b \in \text{range}(f) \to a$ where $a$ is the unique element in $A$ with $f(a) = b$. In particular, when $f$ is bijective (to be defined below), this is the definition of the inverse and the $\bot$ is not used.

bijective, if the function is both surjective and injective.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the function $f(x) = 2x$ defined over the domain $A$ and co-domain $B$ is a bijective function. Can you see why?

Exercise: If $A$ and $B$ are finite sets, and $f : A \to B$ is a bijective function, can you show $|B| = |A|$?

2 Countable Sets

- Definition. A set $S$ is countable if there exists an injective function $f : S \to \mathbb{N}$.

It is called so because the elements of $S$ can be ordered and counted one at a time (although the counting may never finish). More precisely, using $f$ we can devise an ordering of the elements an $S$ and an algorithm which for any natural number $k$ gives the $k$th number in the ordering. Note that since $f$ is injective, for every $n \in \mathbb{N}$, either $f^{-1}(n)$ doesn’t exist, or $f^{-1}(n)$ is a well-defined element in $S$. The following code prints this sequence.

```plaintext
1: for n = 1, 2, 3, \ldots do: \triangleright n \in \mathbb{N} \\
2: if There exists some a \in S such that f(a) = n then: \triangleright i.e. f^{-1}(n) \in S \\
3: Print a
```
• Examples

- **Finite sets** are almost trivially countable. If a set $S$ is finite and $|S| = k$, then the elements of $S$ can be renamed as $\{e_1, e_2, \ldots, e_k\}$. The injective function $f(e_i) = i$ implies $S$ is countable.

Infinite sets can also be countable. $\mathbb{N}$ is countable by definition. But there are many more interesting examples.

- **Set of Integers.** The set $\mathbb{Z}$ is countable. To see this, consider the following function $f : \mathbb{Z} \to \mathbb{N}$. If $x > 0$, then $f(x) = 2x$. If $x \leq 0$, then $f(x) = 2(-x) + 1$. Note that the co-domain of this function is indeed the natural numbers.

For instance, $f(2) = 4$, $f(-2) = 5$, and $f(0) = 1$.

**Claim 1.** The function $f : \mathbb{Z} \to \mathbb{N}$ defined above is injective.

**Proof.** To see this is injective, we need to show $f(x) \neq f(y)$ for two integers $x \neq y$. We may assume, without loss of generality, $x < y$. If both $x$ and $y$ are positive, then $f(x) = 2x < 2y = f(y)$. Similarly, if both are non-negative, then we get $f(x) = -2x + 1 > -2y + 1 = f(y)$. The only other case is $x$ is non-negative and $y$ is positive. In this case, $f(x)$ is odd while $f(y)$ is even. \qed

If we use the above algorithm to figure out the ordering of $\mathbb{Z}$, we get:

$$(0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots)$$