CS 30: Discrete Math in CS (Winter 2019): Lecture 5

Date: 10th January, 2019 (X-Hour)

Topic: More on Countable Sets

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

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1 Countable Sets

We reviewed the definition from yesterday.

- **Closed under Union.**

  Theorem 1. Given two disjoint countable sets A and B, the set $A \cup B$ is countable.

  Proof. Since A is countable, there is a valid, injective function $f : A \to \mathbb{N}$. Since B is countable, there is a valid, injective function $g : B \to \mathbb{N}$. We need to show $A \cup B$ is countable. That is, we need to describe a valid, injective function $h : A \cup B \to \mathbb{N}$. We do so as follows.

  $$\text{For } x \in A \cup B, \;\; h(x) := \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) + 1 & \text{if } x \in B \end{cases}$$

  Note since we assume A and B are disjoint, exactly one of the two cases occur. Furthermore, since f and g maps to natural numbers, the function h also maps to natural numbers. Thus, the definition of h is valid.

  Claim 1. $h : A \cup B \to \mathbb{N}$ is injective.

  Proof. Pick $x, y \in A \cup B$ such that $x \neq y$. We need to show $h(x) \neq h(y)$. There are three cases to consider

  - *Case 1: $x \in A, y \in A$:* In this case $h(x) - h(y) = 2(f(x) - f(y)) \neq 0$ since $f : A \to \mathbb{N}$ is injective.
  - *Case 2: $x \in B, y \in B$:* In this case $h(x) - h(y) = 2(g(x) - g(y)) \neq 0$ since $g : B \to \mathbb{N}$ is injective.
  - *Case 3: $x \in A, y \in B$:* In this case $h(x)$ is even while $h(y)$ is odd, and therefore $h(x) \neq h(y)$.

  Note we don't have to look at the case $x \in B, y \in A$ separately (although it is just one other short line) since we could assume, without loss of generality, that $x \in A, y \in B$ if they lie in different sets. Otherwise we change their names. □

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Remark: *If the above proof looks eerily similar to the proof we did for countability of integers, that is not an accident. The set of integers is the disjoint union of positive numbers (natural numbers), the negative numbers, and the extra* 0. *If we ignore the* 0 *for a moment, then it is indeed the disjoint union of two clearly countable sets.*

Remark: *The above proof really doesn't use disjointness crucially. If* A *and* B *were not disjoint, we could define* $h(x) = 2f(x)$ *if* $x \in A$*, and* $h(x) = 2g(x) + 1$ *if* $x \in B \setminus A$*. Alternately, we could use the fact that* $A \cup B$ *is the disjoint union of* A *and* $B \setminus A$*.*

Remark: *We can use the above argument repetitively to show, for instance, the union of three countable sets is countable, and indeed, the union of* 10^{10} *sets is countable. However, the above* **does not** *imply that if we have infinitely many countable sets* $A_1, A_2, A_3, \ldots$*, then* $\bigcup_{n\in\mathbb{N}} A_n$ *is countable.*

To appreciate this, note that the sum of two rational numbers is rational, and indeed the sum of any finite *collection of rational numbers is* **not necessarily rational**. *Indeed, any irrational number (take your favorite one) has a* non-recurring *decimal representation and is thus a sum of infinite rational numbers. For instance,*

$$\pi = \frac{3}{1} + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \cdots$$

- **The Set of Rationals is Countable.** This may be a surprise since the set of rationals are dense, that is, between any two rational numbers, there is a rational number. Nevertheless, they are countable.

  To show this, we need to construct an injection $g : \mathbb{Q} \to \mathbb{N}$. For now, we only show an injection of $g : \mathbb{Q}_+ \to \mathbb{N}$ where $\mathbb{Q}_+$ are all the positive rationals; we leave the extension to the full set of rationals as an exercise.

  This can be defined as follows: given any positive rational number $z = p/q$ in the *reduced form* (that is, $\gcd(p, q) = 1$), define $g(z) = 2^p 3^q$. Clearly, the functions maps a positive rational number to a positive integer.

  Claim 2. The above function $g : \mathbb{Q}_+ \to \mathbb{N}$ is injective.

  Proof. To see this, pick two different positive rationals $x = p/q$ and $y = r/s$ such that $x \neq y$. We need to prove $g(x) \neq g(y)$. To this end, consider the ratio

  $$\frac{g(x)}{g(y)} = \frac{2^p 3^q}{2^r 3^s} = \frac{2^{p-r}}{3^{s-q}} \tag{1}$$

  Since $x \neq y$, we have $p \neq r$, or $q \neq s$, or both. If $p = r$, then the RHS of (1) is either 1 divided by a positive power of 3, or is a positive power of 3. In neither case, can the RHS be

1. Similarly, if $q = s$, then the RHS of (1) is either a positive power of 2, or 1 divided by a positive power of 2. In neither case can the RHS be 1.

If $p \neq r$ and $q \neq s$, then the RHS of (1) is either a positive power of 2 divided by a positive power of 3, or 1 divided by a positive power of 2 times a positive power of 3, or reciprocals of these. In none of the these cases, can it equate to 1.

In sum, $\frac{g(x)}{g(y)} \neq 1$. Implying $g(x) \neq g(y)$. □

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Exercise: *Extend the above proof to give an injection $g : \mathbb{Q} \to \mathbb{N}$. Hint: use the fact that the union of two countable sets is countable.*

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Exercise: *What ordering of the (positive) rationals does the above give using the algorithm for getting ordering from the injective function? Order the first 7 rationals.*