Functions¹

• **Definition.** A function is a *mapping* from one set to another. The first set is called the *domain* of the function, and the second set is called the *co-domain*. For *every* element in the domain, a function assigns a *unique* element in the co-domain.

Notationally, this is represented as

$$f: A \to B$$

where A is the set indicating the domain dom(f), and B is the set indicating the co-domain codom(f). For every $a \in A$, the function maps the value of $a \mapsto f(a)$ where $f(a) \in B$.

The *range* (sometimes call *image*) of the function is the subset of the co-domain which are *actually* mapped to. That is, $b \in B$ is in the range if and only if there is some element $a \in A$ such that f(a) = b. The range can also be written in the set-builder notation as

$$\operatorname{range}(f) := \{f(a) : a \in A\}$$

• An Example. Suppose

 $A = \{1, 2, 3\}$, and $B = \{5, 6\}$, then the map f(1) = 5, f(2) = 5, f(3) = 6 is a valid function.

A is the domain. B is the co-domain. In this example, B also happens to be the range.

- The Identity Function. When the domain is the same as the co-domain, the *identity* function id : $A \rightarrow A$ maps $a \in A$ to $a \mapsto a$.
- More Examples.
 - Usually (say in calculus) a function is described as a formula like $f(x) = x^2$. Henceforth, whenever you see a function ask your self how does it map to the above definition. In this example, the domain is \mathbb{R} , the set of real numbers, and so is the co-domain. The map is $x \mapsto x^2$ check both are real numbers. The range of the function is the set of non-negative real numbers.
 - $f(x) = \sin x$ is a function whose domain is \mathbb{R} and the range is the interval [-1, 1].
 - A (deterministic) computer program/algorithm is also a function; its domain is the set of possible inputs and its range is the set of possible outputs.

Remark: How about the function $f(x) = \sqrt{x}$? Is this a function? When you think about it, you see some issues if we don't define the domain and co-domain. For instance, if the domain contains negative numbers, then what is $\sqrt{-1}$? Ok, so perhaps the domain is all positive real numbers. However, we also have a problem with $\sqrt{4}$ – is it mapping to +2 or -2? Note it can only map to a unique number. This can be resolved by stating the domain and co-domain are both non-negative reals, and the $x \mapsto \sqrt{x}$ goes to the positive root.

¹Lecture notes by Deeparnab Chakrabarty. Last modified : 16th Jun, 2024

These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

Exercise: Given a set $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, describe a function $f : A \to B$ whose range is $\{5\}$, and describe a function g whose range is $\{4, 6\}$. Just to get a feel, how many functions can you describe of the first form (whose range is $\{5\}$), and how many functions can you describe of the second form?

Pre-Image. Let f : A → B be a valid function. For any element b ∈ B in the co-domain, we define b's pre-image to be the set of elements in the domain which map to b. That is,

$$\mathsf{preimage}(b) := \{a \in A : f(a) = b\}$$

The preimage is a subset of the domain. For example, if $f : A \to B$ where $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ is defined as f(a) = 1, f(b) = 1 and f(c) = 3, then the preimages of the elements in B are

 $preimage(1) = \{a, b\}, preimage(2) = \emptyset, preimage(3) = \{c\}$

- Sur-, In-, Bi- jective functions.
 - A function $f : A \to B$ is called *injective*, if there are no collisions. That is, for any two elements $a \neq a' \in A$, we have $f(a) \neq f(a')$. Such functions are also called *one-to-one* functions.

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f : A \to B$ with f(1) = 5, f(2) = 6, and f(3) = 8. Then, f is injective. This is because f(1), f(2), f(3) are all distinct numbers.

Observe, if $f : A \to B$ is injective, then for any element $b \in B$, |preimage(b)| is either 0 or 1. Injective functions have *inverses*. Formally, given any injective function $f : A \to B$, we can define a function $f^{-1} : range(f) \to A$ as follows

$$f^{-1}(b) = a$$
 where a is the unique $a \in A$ with $f(a) = b$.

- A function f: A → B is called *surjective*, if the range is the same as the co-domain. That is, for every element b ∈ B there exists some a ∈ A such that f(a) = b. Such functions are also called *onto* functions. Alternately, for every b ∈ B, preimage(b) ≠ Ø, and thus |preimage(b)| ≥ 1. For example, if A = {1,2,3} and B = {5,6}, and consider the function f : A → B with f(1) = 5, f(2) = 5, and f(3) = 6. Then, f is surjective. This is because for 5 ∈ B there is 1 ∈ A such that f(1) = 5 and for 6 ∈ B there is a 3 ∈ A such that f(3) = 6.
- *bijective*, if the function is both surjective and injective. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the function f(x) = 2x defined over the domain A and co-domain B is a bijective function. Can you see why?
- Composition of Functions. Given a function *f* : *A* → *B* and a function *g* : *B* → *C*, one can define the composition of *g* and *f*, denoted as *g* ∘ *f* with domain *A* and co-domain *C* as follows:

$$(g \circ f)(a) = g(f(a))$$
 that is $a \mapsto g(f(a))$

Note this is well defined since for every $a \in A$, $f(a) \in B$, and thus $g(f(a)) \in C$. Examples

- Suppose $A = \{1, 2, 3\}$ and $B = \{5, 6\}$ and $C = \{3, 4\}$. Also suppose $f : A \to B$ is defined as f(1) = 5, f(2) = 6, and f(3) = 5; and $g : B \to C$ is defined as g(5) = 3 and g(6) = 4, then the composed function is $(g \circ f)(1) = 3$, $(g \circ f)(2) = 4$, and $(g \circ f)(3) = 3$.
- If $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) = x^2$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $g(x) = \sqrt{x}$ (as defined above), then (convince yourself) that $(g \circ f)(x)$ returns the *absolute* value of x.
- If $f : A \to B$ is a bijection, and $f^{-1} : B \to A$ is its inverse, convince yourself that $(f^{-1} \circ f) : A \to A$ is the id $: A \to A$ identity function.

Some Proofs involving Functions

- Proving a function $f : A \rightarrow B$ is injective/surjective.
 - To prove a function f: A → B is injective, the general principle is to pick two arbitrary different elements in the domain A, call them something like a₁ and a₂, and then use the definition of the function to prove f(a₁) ≠ f(a₂). To give an illustration, consider the function f : N → N mapping n → n². To prove this is injective, fix two different numbers, say m and n, in the domain N. Since they are positive, m + n ≥ 0, and since they are different, m n ≠ 0. Therefore (m + n) · (m n) ≠ 0 (I guess I am using the "fact" that if two numbers multiply to 0 then one of them has to be 0...), that is, m² n² ≠ 0 or, m² ≠ n². So, f(m) ≠ f(n). Done. To prove f : A → B is not injective, all one needs to do is to exhibit two different a₁, a₂ ∈ A with f(a₁) = f(a₂). For instance, if the function was the same as above but the domain was Z, then the function wouldn't be injective since f(1) = f(-1) = 1. Do you see why the above proof (when domain was N) breaks down when the domain is Z?
 - To prove a function f : A → B is surjective, the general principle is to pick an arbitrary element in the domain B, call it b, and use the property of the function to find an element a ∈ A such that f(a) = b. This element a would almost surely depend on b in some way. To prove a function is *not* surjective, one needs to find an element b ∈ B, and then argue that there cannot exist any a ∈ A such that f(a) = b.

For instance, consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which maps $(m, n) \mapsto m + n$. Recall, $\mathbb{N} = \{1, 2, 3, \ldots\}$. We claim that this function is *not* surjective. To see this, we consider the element 1 in the codomain. We now assert there cannot exist any $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that f(m, n) = 1. Indeed, for every $(m, n) \in \mathbb{N} \times \mathbb{N}$, $f(m, n) = m + n \ge 1 + 1 = 2$; we use that every natural number is at least 1.

On the other hand, if the domain of the above function was extended to $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ and the co-domain is also $(\mathbb{N} \cup \{0\})$, then we assert the function does become surjective. Indeed, for any $n \in (\mathbb{N} \cup \{0\})$, we see that $(n, 0) \mapsto n$. Note how the element in the domain, (n, 0)depended on the *n* we chose.

• Relations of sizes of Domain and Co-Domains for Sur/In/Bi-jective functions.

In this bullet point, we are concerned with function $f : A \to B$ where both A and B are *finite* sets, that is, |A| and |B| are some finite numbers. We begin with the following theorem.

Theorem 1. Let $f : A \to B$ be a valid function and let A and B be finite sets. Then,

$$|A| = \sum_{b \in B} |\mathsf{preimage}(b)|$$

Proof. There are two main observations which lead to this.

Observation 1: preimage(b)'s are pairwise disjoint. That is, for any $b \neq b'$ in B, preimage(b) \cap preimage(b') = \emptyset . Why? If not, and suppose there is an element a in both preimages. By definition of the preimage, f(a) = b and f(a) = b' which would contradict the validity of the function.

Observation 2: $A = \bigcup_{b \in B} \text{preimage}(b)$. The latter is a notation of taking the union of all the preimages. First, every preimage is a subset of the domain A, and therefore so is their union. Next, by definition, any element a lies in the preimage of the element f(a), and thus lies in the union of all preimages. Thus, A is a subset of the union of preimages.

The theorem now follows from the claim from previous lecture regarding the cardinality of the union of a finite number of pairwise disjoint sets (asserting its the sum of the individual cardinalities). \Box

Using the above, we can get the following claims.

Claim 1. If A and B are finite sets and $f : A \to B$ is injective, then $|A| \le |B|$.

Proof. We apply Theorem 1 to get

$$|A| = \sum_{b \in B} \quad \underbrace{|\mathsf{preimage}(b)|}_{\leq 1 \text{ since } f \text{ is injective}} \quad \leq \sum_{b \in B} 1 \ = |B|$$

Claim 2. If A and B are finite sets and $f : A \to B$ is surjective, then $|A| \ge |B|$.

Proof. We apply Theorem 1 to get

$$|A| = \sum_{b \in B} \quad \underbrace{|\mathsf{preimage}(b)|}_{\geq 1 \text{ since } f \text{ is surjective}} \quad \geq \sum_{b \in B} 1 = |B|$$

Remark: If A and B are finite sets, and $f : A \to B$ is a bijective function, then |B| = |A|. We will see this useful fact many times in the combinatorics module.

Answers to exercises

• f(1) = 5, f(2) = 5, f(3) = 5 is an example of $f : A \to B$ with range {5}. Similarly, g(1) = 4, g(2) = 4, g(3) = 6 is an example of a function $g : A \to B$ with range {4,6}. There is only one function of the first type. Of the second type there are more. For instanct h(1) = 4, h(2) = 6, h(3) = 6 also has range {4,6}. Can you find one more? How many such functions are there?