Functions[1]

- **Definition.** A function is a *mapping* from one set to another. The first set is called the *domain* of the function, and the second set is called the *co-domain*. For *every* element in the domain, a function assigns a *unique* element in the co-domain.

  Notationally, this is represented as
  $$f : A \to B$$
  where A is the set indicating the domain $\mathrm{dom}(f)$, and B is the set indicating the co-domain $\mathrm{codom}(f)$. For every $a \in A$, the function maps the value of $a \mapsto f(a)$ where $f(a) \in B$.

  The *range* (sometimes call *image*) of the function is the subset of the co-domain which are *actually mapped to*. That is, $b \in B$ is in the range if and only if there is some element $a \in A$ such that $f(a) = b$. The range can also be written in the set-builder notation as
  $$\mathrm{range}(f) := \{f(a) : a \in A\}$$

  > **Remark:** *For any function f with finite domains and ranges, we have $|\mathrm{range}(f)| \leq |\mathbf{dom}(f)|$*

- **An Example.** Suppose
  $$A = \{1, 2, 3\}, \quad \text{and} \quad B = \{5, 6\}, \quad \text{then the map} \quad f(1) = 5, f(2) = 5, f(3) = 6 \text{ is a valid function.}$$
  A is the domain. B is the co-domain. In this example, B also happens to be the range.

- **The Identity Function.** When the domain is the same as the co-domain, the *idenitity* function $\mathrm{id} : A \to A$ maps $a \in A$ to $a \mapsto a$.

- **More Examples.**
  - Usually (say in calculus) a function is described as a formula like $f(x) = x^2$. Henceforth, whenever you see a function ask your self how does it map to the above definition. In this example, this is as follows.
    the domain is $\mathbb{R}$, the set of real numbers, and so is the co-domain. The map is $x \mapsto x^2$ – check both are real numbers. The range of the function is the set of non-negative real numbers (sometimes denoted as $\mathbb{R}_+$).
  - $f(x) = \sin x$ is a function whose domain is $\mathbb{R}$ and the range is the interval $[-1, 1]$.
  - A (deterministic) computer program/algorithm is also a function; its domain is the set of possible inputs and its range is the set of possible outputs.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 7th Jan, 2023*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

Remark: *How about the function $f(x) = \sqrt{x}$? Is this a function? When you think about it, you see some issues if we don't define the domain and co-domain. For instance, if the domain contains* negative *numbers, then what is $\sqrt{-1}$? Ok, so perhaps the domain is all positive real numbers. However, we also have a problem with $\sqrt{4}$ – is it mapping to $+2$ or -2? Note it can only map to a* unique *number. This can be resolved by stating the domain and co-domain are both non-negative reals, and the $x \mapsto \sqrt{x}$ goes to the positive root.*

Exercise: *Given a set $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, describe a function $f : A \to B$ whose range is $\{5\}$, and describe a function g whose range is $\{4, 6\}$. Just to get a feel, how many functions can you describe of the first form (whose range is $\{5\}$), and how many functions can you describe of the second form?*

- **Pre-Image.** Let $f : A \to B$ be a valid function. For any element $b \in B$ in the co-domain, we define b's *pre-image* to be the set of elements in the domain which map to b. That is,

$$\forall b \in B, \ \ \mathsf{preimage}(b) := \{a \in A \ : \ f(a) = b\}$$

The preimage is a subset of the domain. For example, if $f : A \to B$ where $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ is defined as $f(a) = 1$, $f(b) = 1$ and $f(c) = 3$, then the preimages of the elements in B are

$$\mathsf{preimage}(1) = \{a, b\}, \quad \mathsf{preimage}(2) = \emptyset, \quad \mathsf{preimage}(3) = \{c\}$$

Theorem 1. Let $f : A \to B$ be a valid function and let A and B be finite sets. Then,

$$|A| = \sum_{b \in B} |\mathsf{preimage}(b)|$$

Proof. There are two main observations which lead to this.

Observation 1: $\mathsf{preimage}(b)$'s are pairwise disjoint. That is, for any $b \neq b'$ in B, $\mathsf{preimage}(b) \cap \mathsf{preimage}(b') = \emptyset$. Why? If not, and suppose there is an element a in both preimages. By definition of the preimage, $f(a) = b$ and $f(a) = b'$ which would contradict the validity of the function.

Observation 2: $A = \bigcup_{b \in B} \mathsf{preimage}(b)$. The latter is a notation of taking the union of all the preimages. First, every preimage is a subset of the domain A, and therefore so is their union. Next, by definition, any element a lies in the preimage of the element $f(a)$, and thus lies in the union of all preimages. Thus, A is a subset of the union of preimages.

The theorem now follows from the claim from previous lecture regarding the cardinality of the union of a finite number of pairwise disjoint sets (asserting its the sum of the individual cardinalities). □

- **Sur-, In-, Bi- jective functions.**

  – A function $f : A \to B$ is called *injective*, if there are no collisions. That is, for any two elements $a \neq a' \in A$, we have $f(a) \neq f(a')$. Such functions are also called *one-to-one* functions or simply *injections*.

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7, 8\}$, and consider the function $f : A \to B$ with $f(1) = 5, f(2) = 6$, and $f(3) = 8$. Then, f is injective. This is because $f(1), f(2), f(3)$ are all distinct numbers.

Observe, if $f : A \to B$ is injective, then for any element $b \in B$, $|\mathsf{preimage}(b)|$ is either 0 or 1.

Claim 1. If A and B are finite sets and $f : A \to B$ is injective, then $|A| \leq |B|$.

Proof. We apply Theorem 1 to get

$$|A| = \sum_{b \in B} \underbrace{|\mathsf{preimage}(b)|}_{\leq 1 \text{ since } f \text{ is injective}} \leq \sum_{b \in B} 1 = |B|$$

□

Injective functions have *inverses*. Formally, given any injective function $f : A \to B$, we can define a function $f^{-1} : \text{range}(f) \to A$ as follows

$$f^{-1}(b) = a \quad \text{where } a \text{ is the } \textit{unique } a \in A \text{ with } f(a) = b.$$

– A function $f : A \to B$ is called *surjective*, if the range is the same as the co-domain. That is, for every element $b \in B$ there exists some $a \in A$ such that $f(a) = b$. Such functions are also called *onto* functions. Alternately, for every $b \in B$, $\mathsf{preimage}(b) \neq \emptyset$, and thus $|\mathsf{preimage}(b)| \geq 1$.

  For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, and consider the function $f : A \to B$ with $f(1) = 5, f(2) = 5$, and $f(3) = 6$. Then, f is surjective. This is because for $5 \in B$ there is $1 \in A$ such that $f(1) = 5$ and for $6 \in B$ there is a $3 \in A$ such that $f(3) = 6$.

Claim 2. If A and B are finite sets and $f : A \to B$ is surjective, then $|A| \geq |B|$.

Proof. We apply Theorem 1 to get

$$|A| = \sum_{b \in B} \underbrace{|\mathsf{preimage}(b)|}_{\geq 1 \text{ since } f \text{ is surjective}} \geq \sum_{b \in B} 1 = |B|$$

□

– *bijective*, if the function is both surjective and injective.

  For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then the function $f(x) = 2x$ defined over the domain A and co-domain B is a bijective function. Can you see why?

Remark: *If A and B are* finite sets, *and $f : A \to B$ is a bijective function, then $|B| = |A|$. We will see this useful fact many times in the combinatorics module.*

- **Proving a function $f : A \to B$ is injective/surjective.**

  – To prove a function $f : A \to B$ is injective, the general principle is to pick two arbitrary different elements in the domain A, call them something like a_1 and a_2, and then use the definition of the function to prove $f(a_1) \neq f(a_2)$. To give an illustration, consider the function $f : \mathbb{N} \to \mathbb{N}$

mapping $n \mapsto n^2$. To prove this is injective, fix two different numbers, say m and n, in the domain $\mathbb{N}$. Since they are positive, $m + n \geq 0$, and since they are different, $m - n \neq 0$. Therefore $(m + n) \cdot (m - n) \neq 0$ (I guess I am using the "fact" that if two numbers multiply to 0 then one of them has to be 0...), that is, $m^2 - n^2 \neq 0$ or, $m^2 \neq n^2$. So, $f(m) \neq f(n)$. Done.

  To prove $f : A \to B$ is *not* injective, all one needs to do is to exhibit two different $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. For instance, if the function was the same as above but the domain was $\mathbb{Z}$, then the function wouldn't be injective since $f(1) = f(-1) = 1$. Do you see why the above proof (when domain was $\mathbb{N}$) breaks down when the domain is $\mathbb{Z}$?

- To prove a function $f : A \to B$ is surjective, the general principle is to pick an arbitrary element in the domain B, call it b, and use the property of the function to find an element $a \in A$ such that $f(a) = b$. This element a would almost surely depend on b in some way. To prove a function is *not* surjective, one needs to find an element $b \in B$, and then argue that there cannot exist *any* $a \in A$ such that $f(a) = b$.

  For instance, consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which maps $(m, n) \mapsto m + n$. Recall, $\mathbb{N} = \{1, 2, 3, \ldots\}$. We claim that this function is *not* surjective. To see this, we consider the element 1 in the codomain. We now assert there cannot exist any $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $f(m, n) = 1$. Indeed, for every $(m, n) \in \mathbb{N} \times \mathbb{N}$, $f(m, n) = m + n \geq 1 + 1 = 2$; we use that every natural number is at least 1.

  On the other hand, if the domain of the above function was extended to $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ and the co-domain is also $(\mathbb{N} \cup \{0\})$, then we assert the function does become surjective. Indeed, for any $n \in (\mathbb{N} \cup \{0\})$, we see that $(n, 0) \mapsto n$. Note how the element in the domain, $(n, 0)$ depended on the n we chose.

- **Composition of Functions.** Given a function $f : A \to B$ and a function $g : B \to C$, one can define the composition of g and f, denoted as $g \circ f$ with domain A and co-domain C as follows:

$$(g \circ f)(a) = g(f(a)) \quad \text{that is} \quad a \mapsto g(f(a))$$

Note this is well defined since for every $a \in A$, $f(a) \in B$, and thus $g(f(a)) \in C$.

Examples

- Suppose $A = \{1, 2, 3\}$ and $B = \{5, 6\}$ and $C = \{3, 4\}$. Also suppose $f : A \to B$ is defined as $f(1) = 5$, $f(2) = 6$, and $f(3) = 5$; and $g : B \to C$ is defined as $g(5) = 3$ and $g(6) = 4$, then the composed function is $(g \circ f)(1) = 3$, $(g \circ f)(2) = 4$, and $(g \circ f)(3) = 3$.
- If $f : \mathbb{R} \to \mathbb{R}_+$ defined as $f(x) = x^2$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $g(x) = \sqrt{x}$ (as defined above), then (convince yourself) that $(g \circ f)(x)$ returns the *absolute* value of x.
- If $f : A \to B$ is a bijection, and $f^{-1} : B \to A$ is its inverse, convince yourself that $(f^{-1} \circ f) : A \to A$ is the $\mathrm{id} : A \to A$ identity function.

Answers to exercises

- $f(1) = 5, f(2) = 5, f(3) = 5$ is an example of $f : A \to B$ with range $\{5\}$. Similarly, $g(1) = 4, g(2) = 4, g(3) = 6$ is an example of a function $g : A \to B$ with range $\{4, 6\}$. There is only one function of the first type. Of the second type there are more. For instanct $h(1) = 4, h(2) = 6, h(3) = 6$ also has range $\{4, 6\}$. Can you find one more? How many such functions are there?