Graphs: Basics[1]

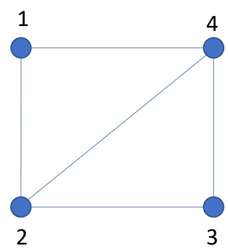
- **What is a graph?**

  A graph $G = (V, E)$ is a pair of sets (both of which will be finite sets for this course). The first set is V called the set of *vertices* or set of *nodes*. The second set E is called the set of *edges* or set of *links*. Elements of E are *unordered pairs* (subsets of size 2) of distinct vertices.

  Example: Say $V = \{1, 2, 3, 4\}$. Say $E = \{(1, 2), (2, 3), (1, 4), (3, 4), (2, 4)\}$. Then $G = (V, E)$ is one graph on 4 vertices and 5 edges.

  Remark: *Although every element of E is a subset itself, instead of using curly braces we use parenthesis. This is just convention, and in my mind it distinguishes an edge from an arbitrary pair of vertices. Some textbooks (especially ones on graph theory) use curly braces for edges.*

  Pictorial Representation. Almost everyone I know thinks about graphs *pictorially*. The vertices of the graph are drawn as points/circles on the plane. The edges are drawn as straight (or sometime non-straight) lines. For example, the graph above is pictorially represented as follows



  Remark: *The pictures are good for intuition. The final proof however, as you know, should never be using a picture. It should be formal, and I'll try to give proofs with only words. A picture is okay for illustration, not demonstration.*

- **Why graphs?** Graphs are amazing objects to argue about things which have pairwise relations between them. Perhaps the graph which affects all our lives is the **Web Graph**. The nodes are all web-pages in the world; there is an edge between two web-pages if they link each other. Then there is the **Social/Facebook Graph**. The nodes are individuals; there is an edge between two nodes if they are friends.

  But graphs come up every where. Molecules are often modeled as graphs in computational biology. Agents and Items they wish to purchase are often modeled as graphs in economics. Processors and Jobs are modeled as graphs in scheduling. The list is endless, and **Graph Theory** is an extremely important object of study.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

- **Notations.** There is a lot of notation in graph theory; but they are often picturesque and intuitive. One of the goals of this module is to actually acquaint you of these. Below we fix a graph $G = (V, E)$.
  - Given an edge $e = (u, v)$, the vertices u and v are the ***endpoints*** of e. We say e ***connects*** u and v. We say that u and v are ***incident*** to e.
  - Two vertices $u, v \in V$ are ***adjacent*** or ***neighbors*** if and only if (u, v) is an edge.
  - The ***incident edges*** on v is denoted using the set $\partial_G(v)$. So,

    $$\partial_G(v) := \{(u, v) : (u, v) \in E\}$$

    We lose the subscript if the graph G is clear from context.
  - Given a vertex v, the ***neighborhood*** of v is the set of neighbors of v. This is denoted sometimes as $N_G(v)$ or sometimes as $\Gamma_G(v)$. So,

    $$N_G(v) := \{u : (u, v) \in E\}$$

    We lose the subscript if the graph G is clear from context.
  - The cardinality of $N_G(v)$ is called the ***degree*** of vertex v. We denote it using $\deg_G(v)$. This counts the number of neighbors of v. Note that,

    $$\deg_G(v) = |N_G(v)| = |\partial_G(v)|$$

  - A vertex v is ***isolated*** if its degree is 0. That is, it has no edges connected to it.
  - A graph $G = (V, E)$ is called ***regular*** if all degrees are equal, that is, $\deg_G(v) = \deg_G(u)$ for all u and v.
  - Given a graph $G = (V, E)$, we use $V(G)$ to denote V and $E(G)$ to denote E. This notation is useful when we are talking about multiple graphs.

- **Deleting and Inserting Edges and Vertices from a graph.**

  Fix a graph $G = (V, E)$. Let $e = (u, v)$ be an edge in E. We get a new graph by ***deleting*** the edge e from G. This graph is denoted as $G - e$ or $G \setminus e$. $V(G \setminus e) = V$ and $E(G \setminus e) = E \setminus e$.

  $$G - e := G \setminus e := (V, E \setminus e)$$

  Note $|V(G \setminus e)| = |V(G)|$ but $|E(G \setminus e)| = |E(G)| - 1$.

  Given a susbet $F \subseteq E$ of edges, we can delete all the edges in F iteratively to get the graph $G - F$ (this is not a usual notation). In particular, we get the graph G' defined as $G' = (V(G), E(G) \setminus F)$.

  Similarly, we can *add/insert* edges to G. Let $e = (u, v)$ be a pair of vertices. Then, we get a new graph by *inserting* the edge e in G. This graph is denoted as $G + e$ or $G \cup e$. Note if e was already present in E, then $G + e = G$.

  $$G + e := G \cup e := (V(G), E(G) \cup e)$$

  We can also delete a *vertex*. When we delete a vertex, we delete that vertex from the vertex set and *also* all the edges adjacent to v. This new graph is called $G - v$ or $G \setminus v$.

  $$G - v := G \setminus v = (V(G) \setminus v,\ E(G) \setminus \partial(v))$$

  Note that $|V(G - v)| = |V(G)| - 1$ and $|E(G - v)| = |E(G)| - \deg_G(v)$. Note that $|E(G - v)|$ may be equal to $|E(G)|$; this occurs if v was an isolated vertex in G.

- **Subgraphs and Induced Subgraphs.**

  A graph $H = (W, F)$ is a ***subgraph*** of a graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$, and if (W, F) is a valid graph. That is, for any edge $(u, v) \in F$, both u and v are in the set W

  Given a graph $G = (V, E)$ and a subset $W \subseteq V$ of vertices, the ***induced subgraph*** $G[W] = (W, F)$ where $F \subseteq E$ and *any* original edge $(u, v) \in E$ with both endpoints $u, v \in W$ lies in F.

- **The Handshake Lemma.** The first proof in graph theory is something you have already seen before in a UGP.

  Theorem 1. For any graph $G = (V, E)$, we have

  $$\sum_{v \in V(G)} \deg_G(v) = 2|E(G)| \tag{1}$$

  Proof. We will give "three" proofs of the above theorem. The last two are basically the same, although they may feel different. The first is based on the combinatorial idea of showing that the LHS and RHS actually count the same set.

  Proof 1. Given the graph $G = (V, E)$, consider the following set

  $$X = \{(e, v) : e \in E, v \in V, v \in e\}$$

  That is, X is a set of tuples whose first entry is an edge in G, and the second entry is a vertex incident on that edge.

  One way of counting X is using the product rule from left to right. There are $|E(G)|$ choices for the first index e. Given any such e, there are 2 choices of vertices incident on it. Therefore,

  $$|X| = 2|E(G)|$$

  Next, we count the set X from right to left. Here, we first pick a vertex v. Then, given any v, how many edges e can go in the first coordinate? We see that all such e's must be incident on v. The number is precisely $\deg_G(v)$. We cannot apply the product rule as the degrees could be different for different vertices. But we just apply the sum rule, vertex-per-vertex. Formally, given any vertex $a \in V$, we define the set

  $$X_a := \{(e, v) : e \in E(G), v \in V(G), v \in e, \quad \text{and} \quad v = a\}$$

  Note, $X_a \cap X_b = \emptyset$ for distinct vertices a and b. Note $X = \bigcup_{a \in V} X_a$ (union of X_a's with a going over all vertices), and thus, $\sum_{a \in V} |X_a| = |X|$. FInally, for a fixed vertex $a \in V$, we get $|X_a| = \deg_G(a)$. Thus, we get

  $$|X| = \sum_{a \in V(G)} \deg_G(a)$$

  completing the proof.

  Proof 2. Next, we see a proof by induction. Induction on graphs is something we need to get used to. Fast.

Let $P(m)$ be the predicate that is true if *for all* graphs $G = (V, E)$ with $|E| = m$, the equality (1) holds. Note that $P(m)$ itself is a predicate logic statement. Also note that the theorem statement is asking us to prove that $\forall m \in \mathbb{N} \cup \{0\} : P(m)$ is true. We add the $\{0\}$ because there can be graph with 0 edges ($E(G) = \emptyset$). (This kind of predicate should remind you of predicates we defined for proving correctness of code.)

Base Case: Is $P(0)$ true? We need to show for all graphs $G = (V, E)$ with $|E| = 0$, (1) holds. If G has no edges, then the RHS equates to 0. However, no edges also implies $\deg_G(v) = 0$ for all $v \in V(G)$. Thus, the LHS of (1) is also 0. The base case is established.

Inductive Case: Fix a natural number $k \geq 0$. Assume $P(k)$ is true. We need to show $P(k+1)$ is true.

Once again, let us spell out what we have assumed and what we need to show. We have assumed that *for all* graphs with k edges, the equality (1) is true. We need to show *for any* graph with $k+1$ edges, the equality (1) holds. To this end, fix a graph $G = (V, E)$ with $|E| = k + 1$.

Let (u, v) be an *arbitrary* edge in G. Consider the graph $G' := G - (u, v)$ (recall the definition of deletion of an edge). By definition of deletion, we have the following,

$$\deg_{G'}(w) = \begin{cases} \deg_G(w) - 1 & \text{if } w = u \text{ or } w = v \\ \deg_G(w) & \text{otherwise} \end{cases} \tag{2}$$

Now, note by Induction Hypothesis, since $|E(G')| = k$ and $P(k)$ is true, we get

$$\sum_{w \in V(G')} \deg_{G'}(w) = 2|E(G')| \tag{IH}$$

Finally, note

$$|E(G')| = |E(G)| - 1 \tag{3}$$

Now we have all the ingredients to complete the proof. We get

$$\begin{aligned} \sum_{v \in V(G)} \deg_G(v) &= \sum_{w \in V(G)} \deg_{G'}(w) + 2 && \text{Only } u \text{ and } v\text{'s degrees are different} \\ &= 2|E(G')| + 2 && \text{Only (IH)} \\ &= 2\left(|E(G)| - 1\right) + 2 && \text{Using (3)} \\ &= 2|E(G)| \end{aligned}$$

We established $P(k+1)$ and thus, by induction, $P(m)$ is true for all $m \in \mathbb{N} \cup \{0\}$. That is, the theorem holds.

Proof 3. We will re-cast the above induction proof into a proof which looks at a *minimal counter-example*. This is a method which is similar to induction, but may help some picture what is happening better. Use whatever works. The proof goes as follows.

Suppose, for contradiction's sake, the theorem is false, and there exist graphs for which (1) is not true. Among all such *counterexample* graphs, pick a G with the smallest number of edges.

Observe, $|E(G)| > 0$, since a graph with zero edges does satisfy (1) (this argument is the same as the base case of the induction proof).

Let (u, v) be any edge in G, and consider $G' := G - (u, v)$. Since G' has *strictly fewer edges* than in G, we see that (1) must hold for G' since G was the counterexample with the smallest number of edges. Thus, we get (IH) is true (the induction hypothesis). The rest of the argument is similar to one above, and this shows (1) holds for G as well. Which is a contradiction to the fact that G was a counterexample.

Where the above proof "wins" is that one doesn't need to define a predicate, etc. But it is the same induction proof. See more details in the supplementary lecture notes of the induction lectures. $\square$