Graphs: Bipartite Graphs[1]

- **Bipartite Graphs.**

  A graph $G = (V, E)$ is ***bipartite*** if there is partition of $V = L \cup R$ such that $L \cap R = \emptyset$ and for every edge $e = (u, v) \in R$, we have $|\{u, v\} \cap L| = |\{u, v\} \cap R| = 1$. That is, every edge has exactly one endpoint in L and exactly one endpoint in R.

  Remark: *Note this means that for any two vertices in L, there are no edges between them. For any two vertices in R, there are no edges between them.*

  A bipartite graph is often written with the explicit partition, that is, it is often written as $G = (L \cup R, E)$ which shows the partition of the vertex set into L and R.

- **Why?** Bipartite graphs are very useful objects to denote relations between two classes of objects: agents-items, jobs-machines, students-courses, etc.

  For example, suppose we have a set of students and a set of offered classes. Each student wants to take a certain subset of classes. All this can be captured in a bipartite graph. Let S be the set of students and C be the set of classes. Consider the graph $G = (S \cup C, E)$ (it is bipartite by the notation), where we have an edge from a student $s \in S$ to a class $c \in C$ if s wants to be in the class c.

  It is also an important class of graphs in that some problems (and we will see one of them soon) is just easier to solve on bipartite graphs. Thus, it would be nice to know when a given graph $G = (V, E)$ is indeed bipartite. The following theorem shows a characterization.

- **Characterization.**

  Theorem 1. A graph $G = (V, E)$ is bipartite if and only if it has no odd-length cycles.

  Remark: *Before we prove the above theorem, let me give a philosophical interpretation of this, which is actually pretty deep. Let's see if something gets across.*

  Imagine an all powerful person, let us call them Prover *who just knows whether or not a graph G is bipartite or not. So, if we have a graph G, we can go to* Prover *and get the answer. But we* Verifiers *are skeptics; we need a* reason *as well for the answer. A reason we can verify. So we ask* Prover*, "Hey, give me some evidence for your answer." Now let us wonder what evidence* Prover *can give.*

  If G was indeed bipartite, then Prover *explicitly tells us what the partition L and R is. Then we as* Verifiers *can go over edges of our graph G and check that exactly one end point lies in the part L and the other in the other part R. And then we will be convinced.*

  But what if G is not bipartite? Naively, Prover *would have to go over* all *partitions $L \cup R$, and for each of them exhibit an edge (u, v) whose both endpoints lie in L or R. There are 2^n partitions of*

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These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

an n-vertex graph (right?). Reading and verifying this evidence, for a 1000 vertex graph, would take us centuries. So no, not satisfactory.

Here is where the above characterization helps! The above theorem asserts that if G is not *bipartite, then G must have an odd-length cycle C. This is what* Prover *gives us. Upon receiving, we check that the cycle indeed is present in G, and then we are convinced that indeed G is not bipartite (see Lemma 1). And all is good.*

*All the stuff above might seem like a fun(?) story, but there are actual real-world parallels today. Most computing is done on the cloud today by some entities who have lot of compute power (*Prover*s), and we verifiers send this to them. However, they also must send us back proofs/evidence. If we sent them some data to do some analytics, they should also give us reasons for their various hypothesis. As of today, most of them don't. And, well, it is a big issue. This forms one part of the burgeoning area called "Interpretable AI".*

Ok enough of a digression, moving to stuff we can prove.

We prove this theorem, piece by piece, over three lemmas. The first is a necessary condition

Lemma 1. If G has an odd-length cycle, then G is not bipartite.

Proof. Suppose, for contradiction's sake, G is bipartite. That is, there exists disjoint subsets L and R such that $V(G) = L \cup R$ and every edge $(u, v) \in E(G)$ has one endpoint in L and one endpoint in R.

Now, let $C = (x_1, x_2, \ldots, x_{2k}, x_{2k+1})$ be the odd cycle, for some $k \geq 1$. The first vertex x_1 lies in L or R. Without loss of generality assume $x_1 \in L$ (otherwise swap the names of L and R). This implies $x_2 \in R$ since $(x_1, x_2) \in E$, $x_3 \in L$, $x_4 \in R$, and so on. More generally, $x_{2i+1} \in L$ and $x_{2i} \in R$ for any natural i. But this leads to $x_{2k+1} \in L$. This is a contradiction since $(x_{2k+1}, x_1) \in E(G)$, and $x_1 \in L$ as well. □

So we have shown that if a graph G contains an odd cycle, it is not bipartite. The more interesting direction is to prove if G does *not* contain an odd cycle, then it must be bipartite. To do so, first we consider the case of graphs with *no* cycles (that is, forests). Clearly, a graph with no cycles contains no odd cycles. The fact better be true for these.

Lemma 2. A forest $G = (V, E)$ is bipartite.

Proof. We prove the lemma by induction on the number of vertices. Let $P(n)$ be the predicate which takes the value true if every forest $G = (V, E)$ with $|V| = n$ is bipartite. We wish to show $\forall n \in \mathbb{N} : P(n)$ is true.

Base Case: $P(1)$ is true. Indeed any graph with 1 vertex (and therefore no edges) is bipartite trivially; $L = V, R = \emptyset$.

Inductive Case: Fix a natural number k, and assume $P(k)$ is true. We need to show $P(k+1)$ is true. Once again, this means every forest on k vertices is bipartite, and we need to prove every forest on $(k+1)$-vertices is bipartite. To this end, fix a forest $G = (V, E)$ with $|V| = k + 1$. We know from the previous lectures that there must exist a $v \in V$ with $\deg_G(v) \leq 1$. Consider the graph

$G' = G - v$. Note two things. One, G' is a forest since deleting an edge cannot introduce cycles. Two, $|E(G')| \geq |E(G')| - 1$ since $\deg_G(v) \leq 1$.

Since $|V(G')| = k$ and G' is a forest, by the induction hypothesis we get G' is bipartite. That is, there exists a partition $L \cup R$ of $V(G')$ such that for any edge $(x, y) \in E(G')$, both x and y are not in the same partition. If $\deg_G(v) = 0$, add v arbitrarily to any part. If $\deg_G(v) = 1$ and if $(u, v) \in E(G)$ is the unique edge incident to v in G, then if $u \in L$ add v to R, and otherwise add v to L. This is a valid bipartition of the vertices of G. This prove $P(k + 1)$ is true, and by induction the statement is true. □

Good, at least graphs with *no* cycles are bipartite. Now we have all the ingredients for the proof of Theorem 1.

Proof of Theorem 1. We proceed to prove this by induction as well. To this end, let $P(m)$ be the predicate which takes the value true if any graph with exactly m edges and no odd cycles is bipartite. We need to show that $\forall m \in \mathbb{N} : P(m)$ is true.

> **Base case**: We prove $P(1)$ is true. Let G be an arbitrary graph with $|E(G)| = 1$. Let this edge by (x, y). Then, let $L = \{y\}$ and $R = V(G) \setminus y$. Since every edge of G (there is only one) has exactly one end point in L and the other in R, this proves G is bipartite. (PS: We also could have just used Lemma 2 since a graph with one edge must be a forest.)
>
> **Inductive Case**: Let $k \geq 1$ and assume that $P(k)$ is true. We need to show $P(k + 1)$ is true. To this end, fix an arbitrary graph $G = (V, E)$ with *no odd cycles* and with $|E| = k + 1$.
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> If G has no cycles, then by Lemma 2, G is bipartite. Therefore, henceforth we assume that G has at least one cycle. Let $C := (u_1, \ldots, u_\ell, u_1)$ be any cycle in G. Note ℓ must be *even*.
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> Now consider the graph $G' = G - (u_\ell, u_1)$ formed by deleting a single edge. All cycles in G' have even length (since all cycles in G have even length and no new cycles have been introduced), and $|E(G')| = k$. So, by the induction hypothesis, G' is bipartite. Let (L, R) be a bipartition of G'.
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> Now, note that $(u_1, \ldots, u_\ell)$ is a path in G'. Without loss of generality assume $u_1 \in L$ (otherwise swap the names). Therefore, we get $u_2 \in R$, $u_3 \in L$, and so on, more generally, $u_{2i+1} \in L$ and $u_{2i} \in R$. In particular, $u_\ell \in R$ since ℓ is even.
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> Therefore, (L, R) is also a bipartition of G. Indeed, (u_1, u_ℓ) satisfies the bipartition condition. And so does every edge $e \in E(G')$. Thus G is bipartite implying $P(k + 1)$ is true.

The theorem follows from induction. □