Graphs: Proof of Hall's Theorem[1]

- **Recap.**

  A graph $G = (V, E)$ is ***bipartite*** if there is partition of $V = L \cup R$ such that $L \cap R = \emptyset$ and for every edge $e = (u, v) \in R$, we have $|\{u, v\} \cap L| = |\{u, v\} \cap R| = 1$. That is, every edge has exactly one endpoint in L and exactly one endpoint in R.

  A ***matching*** M in a graph is a subset of edges $M \subseteq E$ such that for any $e, e' \in M$, $e \cap e' = \emptyset$. That is, M is a collection of edges which do not share end points. A vertex $v \in V$ participates in the matching M if there is an edge in M which is incident to v. In a bipartite graph $G = (L \cup R, E)$, a matching $M \subseteq E$ is an L-matching if all vertices in L participate in M.

- **Hall's Theorem** Given any subset $S \subseteq L$, we $N_G(S)$ are the set of vertices in R which neighbors of some vertex in S. Hall's Theorem says the following.

  Theorem 1. Let $G = (V, E)$ be a bipartite graph with $V = L \cup R$. Then, G has an L-matching if and only if

  $$\text{For every subset } S \subseteq L, \ |N_G(S)| \geq |S| \qquad \text{(Hall's Condition)}$$

  Proof. Again, one direction is easy. That is, if $G = (L \cup R, E)$ has an L-matching, then we must have (Hall's Condition). Why? Suppose there exists an L-matching called M. Then for any $S \subseteq L$, consider the set $T = \{v \in R : \exists u \in S : (u, v) \in M\}$. That is, look at all the partners in M, of vertices in S. Clearly, $T \subseteq N_G(S)$, and thus, $|N_G(S)| \geq |T|$. And $|T| = |S|$ since every vertex in S has a partner in M (M is an L-matching). So, $|N_G(S)| \geq |S|$.

  The interesting direction is the converse. Given that (Hall's Condition) holds, we need to prove that $G = (L \cup R, E)$ has an L-matching. We will prove by induction **on vertices**. This proof is deep, in that it has layers. So hold tight!

  > Let $P(n)$ be the predicate which is true if any bipartite graphs $G = (L \cup R, E)$ with $|L| = n$ satisfying (Hall's Condition) has an L-matching.

  We need to show $\forall n \in \mathbb{N} : P(n)$ is true; we proceed to prove this by induction.

  Base Case: Is $P(1)$ true? Fix any graph $G = (L \cup R, E)$ with $|L| = 1$. Let $L = \{v\}$. (Hall's Condition) implies, $\deg_G(v) \geq 1$. So, there is some edge (v, w) incident on v. $M = \{(v, w)\}$ is an L-matching. So, $P(1)$ is true.

  Inductive Case: Fix a natural number k. We assume $P(1), P(2), \ldots, P(k)$ are all true. We wish to prove $P(k+1)$. To that end, we fix a bipartite graph $G = (L \cup R, E)$ which satisfies (Hall's Condition) and $|L| = k + 1$.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

Let $u \in L$ be an arbitrary vertex. (Hall's Condition) implies $\deg(u) \geq 1$, thus there is at least one edge $(u, v) \in E$. Pick one such edge *arbitrarily*. Consider the graph $G' = G - \{u, v\}$. That is, we delete both **vertices** u and v (and not just the edge (u, v)). G' is also a bipartite graph, with $G = (L' \cup R', E')$ where $L' = L - u$, $R' = R - v$ and $E' = E \setminus (N_G(u) \cup N_G(v))$. See Figure 1 for an illustration.

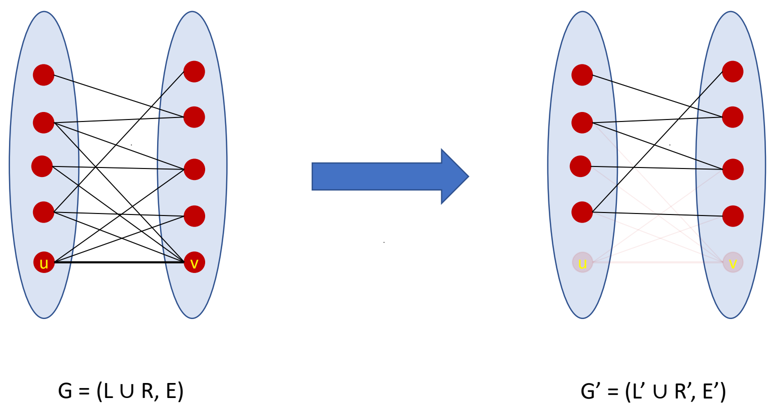


Figure 1: Deleting the vertices u and v.

We now fork into two cases.

Case 1: G' satisfies (Hall's Condition). This is the easy case. Since $|L'| = |L| - 1 = k$, and since by the induction hypothesis, $P(k)$ is true, we get that G' has an L'-matching; let's call it M'. Then, $M := M' \cup (u, v)$ is the required L-matching in G. So in this case, we have proven $P(k + 1)$.

Case 2: G' doesn't satisfy (Hall's Condition). What does this mean? It means there is some subset $S \subseteq L'$, such that $|N_{G'}(S)| < |S|$. On the other hand, since G did satisfy (Hall's Condition), we have $|N_G(S)| \geq |S|$. Finally, note that the only way $N_{G'}(S)$ and $N_G(S)$ can be different is that if $N_G(S)$ has the vertex v in it. And in that case, $N_{G'}(S) = N_G(S) \setminus v$. See Figure 2 for an illustration.

Therefore, we have $v \in N_G(S)$ and furthermore, $|N_G(S)| = |S|$; if $|N_G(S)| > |S|$, then indeed, $|N_G(S)| \geq |S| + 1$ because the LHS is an integer, which in turn implies $|N_{G'}(S)| = |N_G(S)| - 1 \geq |S|$.

Now, we consider two different graphs. We consider $G_1 = G[S \cup N_G(S)]$ and $G_2 = G[(L \setminus S) \cup (R \setminus N_G(S))]$. Recall, the notion of induced subgraphs. See Figure 3 for an illustration.

Claim 1. Both G_1 and G_2 satisfy (Hall's Condition).

Proof. Let's first prove for G_1. Any subset $T \subseteq S$ has $N_G(T) \subseteq N_G(S)$. Thus, $N_{G_1}(T) = N_G(T)$ as well. Since G satisfied (Hall's Condition), we get $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. Thus, G_1 satisfies (Hall's Condition).

Moving on to G_2. Fix a subset $T \subseteq L \setminus S$. What is $N_{G_2}(T)$? Here is an useful observation:

$$N_{G_2}(T) \;=\; N_G(T) \setminus N_G(S) \;=\; N_G(S \cup T) \setminus N_G(S)$$

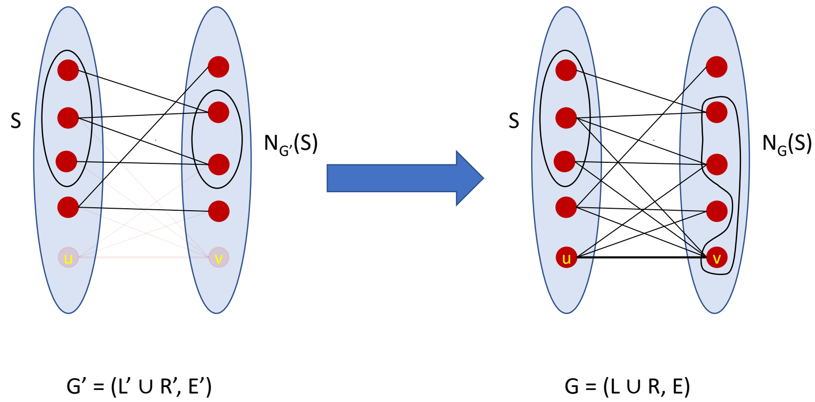


Figure 2: How to related $N_{G'}(S)$ and $N_G(S)$.

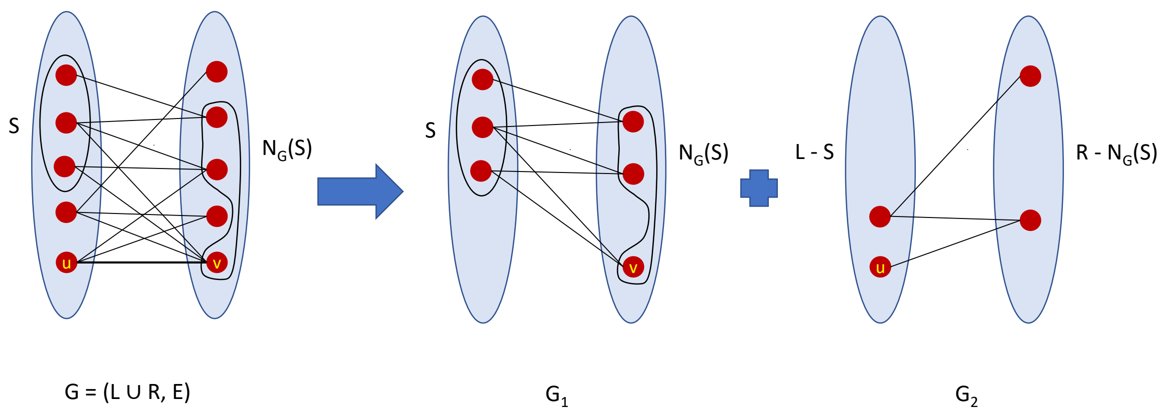


Figure 3: Breaking into two graphs.

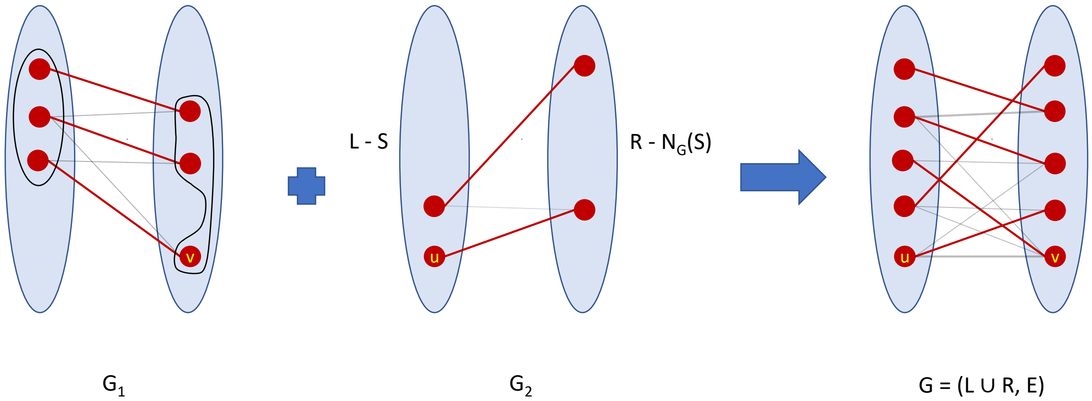
The first equality follows since the neighbors of T in G_2 are precisely the neighbors of T in G which are not the neighbors of S in G. The second equality is the clever part; it is noting that even if we look at neighbors of $S \cup T$ in G and remove the neighbors of S, we still get the neighbors of T in G which are not in $N_G(S)$. Why is this useful? Because, $N_G(S) \subseteq N_G(S \cup T)$. Thus, we know that $|N_G(S \cup T) \setminus N_G(S)| = |N_G(S \cup T)| - |N_G(S)|$.

Putting all together, we get

$$|N_{G_2}(T)| = |N_G(S \cup T)| - |N_G(S)| \geq |S \cup T| - |S| = |T|$$

where the inequality follows since $|N_G(S \cup T)| \geq |S \cup T|$ by (Hall's Condition) and since $|N_G(S)| = |S|$, and the second equality follows since $S \cap T = \emptyset$. $\square$

Since both G_1 and G_2 satisfy (Hall's Condition), and since both $|S|$ and $|L \setminus S|$ are $< |L|$, by the induction hypothesis, we get that G_1 has an S-matching called M_1 and G_2 has an $L \setminus S$-matching called M_2. Thus, $M_1 \cup M_2$ is the L-matching in G.



Done! □