Principle of Mathematical Induction[1]

- **Formal Setting.**

  Mathematical Induction is used to prove theorems of the form $\forall n \in \mathbb{N} : P(n)$ where P is some predicate with the natural numbers as the domain of discourse. Formally, it is stated as follows

$$\Big(P(1) \wedge (\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1))\Big) \Rightarrow (\forall n \in \mathbb{N} : P(n)) \qquad \text{(PMI)}$$

  In plain English, it asserts that to prove the statement "$P(n)$ is true for all $n \in \mathbb{N}$.", it suffices to prove

  - **The Base Case:**(*often easy*) Prove that $P(1)$ is true; and
  - **The Inductive Case:**(*the meat!*) For any natural number k, if $P(k)$ is true, then prove that $P(k+1)$ is true.

  In the inductive case, the *assumption* that "$P(k)$ is true" is called the Induction Hypothesis.

- **Arithmetic Series**

  Theorem 1. For all positive integers n, $\sum_{i=1}^{n} i = n(n+1)/2$

  The predicate $P(n)$ takes the value true if $\sum_{i=1}^{n} i = n(n+1)/2$ and false otherwise. Theorem 1 asserts that $P(n)$ is true for all natural numbers.

  Proof. To prove $\forall n \in \mathbb{N} : P(n)$, the principle of mathematical induction (or simple induction, henceforth) asks us to check/prove the following.

  Base Case: Let us verify that $P(1)$ is true. Indeed, $\sum_{i=1}^{1} i = 1$ and $\frac{1(1+1)}{2} = 1$, and thus $P(1)$ is true.

  Inductive Case: Fix any natural number k. The induction hypothesis is that $P(k)$ is true. We need to prove $P(k+1)$ is true.

  $P(k)$ is true implies

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \qquad \text{(Induction Hypothesis)}$$

  To prove $P(k+1)$ is true, that is, we need to show

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \qquad \text{(Need to Show)}$$

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 18th Jan, 2023*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

We establish this by noting that the LHS of (Need to Show) is

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1) \cdot \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

where in the second inequality we have used the (Induction Hypothesis). Thus, we have established (Need to Show), and thus $\forall n \in \mathbb{N} : P(n)$ follows from induction. □

Exercise: *Using induction, prove* $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ *for any integer* $a > 1$ *and non-negative integer* n*.*

- **A Divisibility Fact.** We now prove the following fact by induction.

Theorem 2. For all $n \in \mathbb{N}$, 3 divides $n^3 - n$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 3 divides $1^3 - 1 = 0$. Indeed, 3 times 0 is 0.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 3 divides $k^3 - k$. We need to show $P(k+1)$ is true, that is, 3 divides $(k+1)^3 - (k+1)$. To do so, we expand $(k+1)^3$, to get

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$$

$3(k^2 + k)$ is divisible by 3, and by the *induction hypothesis* (that is, $P(k)$ is true), $k^3 - k$ is divisible by 3. Therefore, $(k+1)^3 - (k+1)$ is divisible by 3. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Exercise: *Does* 4 *divide* $n^4 - n$ *for all non-negative integers* n*? Mimic the above proof.*

- **Another Divisibility Fact.** We now prove the following fact by induction.

Theorem 3. For all $n \in \mathbb{N}$, 7 divides $3^{2n} - 2^n$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 7 divides $3^2 - 2^1 = 7$. Indeed it does. Therefore $P(1)$ is true.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 7 divides $3^{2k} - 2^k$. We need to show $P(k+1)$ is true, that is, 7 divides $3^{2(k+1)} - 2^{(k+1)}$. Indeed observe,

$$\begin{aligned} 3^{2(k+1)} - 2^{(k+1)} &= 3^2 \cdot 3^{2k} - 2 \cdot 2^k \\ &= 9 \cdot 3^{2k} - 2 \cdot 2^k \\ &= 7 \cdot 3^{2k} + 2 \cdot 3^{2k} - 2 \cdot 2^k \\ &= 7 \cdot 3^{2k} + 2 \cdot \left(3^{2k} - 2^k\right) \end{aligned} \tag{1}$$

7 divides $3^{2k} - 2^k$, by the induction hypothesis. 7 clearly divides $7 \cdot 3^{2k}$. Therefore, 7 divides $3^{2(k+1)} - 2^{k+1}$. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

This proof was (slightly) tricky. Line (6) is where the trick was where we split the 9 as $7+2$. Why did we do that? Well, we knew something about $3^{2k} - 2^k$, but when we expanded out we got $9 \cdot 3^{2k} - 2 \cdot 2^k$. If the "coefficients" of 3^{2k} and 2^k were same we would be done (but it isn't), so we just went ahead and did that. It seems like a happy accident, but that is by design: the person who devised this theorem (in this case, me) probably worked backwards to come up with the statement. □

Exercise: *Can you come up with statements like above? Can you guess which number will always divide $4^{3n} - 3^{2n}$ for all natural numbers n? After guessing, can you prove that guess using induction.*

Remark: *Sometimes, the induction principle may look as follows: (a) The base case may involve proving $P(1), P(2), \ldots, P(c)$ for some finite c, and (b) The inductive case may be possible only for numbers $k \geq c$. Note this is also perfectly OK to establish $\forall n : P(n)$. We will see such an example in class and problem sets.*

- **A Geometry Fact.** We now look at an example of a proof by induction where the predicate $P(n)$ is itself a statement in predicate logic. The statement we want to prove is one from high-school geometry which you may have seen before. Below, an n-gon is a polygon on the plane with n vertices and edges.

Theorem 4. The angles of any ***convex*** n-gon for $n \geq 3$, measure in radians, adds up to $(n-2)\pi$.

Before we begin, we need to recall/define what a *convex* n-gon is. For our purposes, it would suffice to say a n-gon is convex if joining any two vertices which are two-apart by a line segment splits the vertices into two opposite sides of it. More precisely, if we number the vertices in counter-clockwise order as $p_1, p_2, \ldots, p_n$, then if we join p_i and p_{i+2} by a line segment, the vertex p_{i+1} is on one side, and all the other vertices are on the other side of this line segment. By "side", we mean that if we extend the line segment to an infinite line, it partitions the plane into two sides (or halves). See Figure 1 for an illustration. Clear? It's a bit of jargon, but let's move on to the point I want to make.

Proof. We want to proceed by induction. But let's be clear what the predicate is. Remember, we want the predicate $P(n)$ to depend only on a natural number n. What should it be? It is this. $P(n)$ is true iff **every** convex polygon on n vertices has its angles adding up to $(n-2)\pi$. That is, $P(n)$ itself is a statement of the form $\forall \cdots$. More formally, here it is

$$P(n) = \forall \text{ convex polygon } (p_1, \ldots, p_n) \quad : \underbrace{\angle p_n p_1 p_2 + \sum_{i=2}^{n-1} \angle p_{i-1} p_i p_{i+1} + \angle p_{n-1} p_n p_1}_{\text{sum of angles of the } n\text{-gon}} = (n-2)\pi$$

We want to prove $\forall n \in \mathbb{N}, n \geq 3 \; : \; P(n)$ is true.

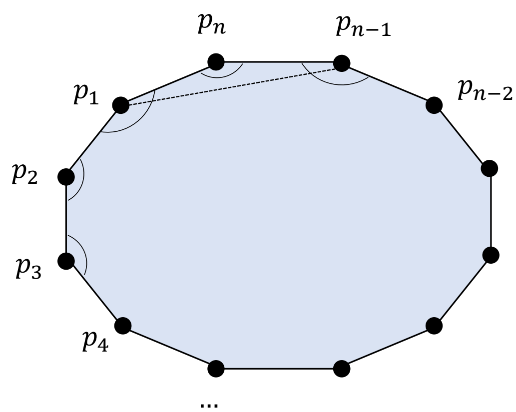

Figure 1: A Convex n-gon. Note that joining p_{n-1} and p_1 splits the n-gon into a triangles on one side and a convex $(n-1)$-gon on the other.

Base Case. We start with $n = 3$. The statement $P(3)$ is true if ***every*** triangle (all triangles are convex) has its angles sum up to π. This is a fact which is true in Euclidean geometry. We will not prove this here, but we will just bring to attention that this statement $P(3)$ is itself a statement about all triangles.

Inductive Case. We fix a $k \geq 3$ and assume $P(k)$ is true. That is, we assume

$$\forall \text{ convex polygon } (p_1, \ldots, p_k) \quad : \angle p_k p_1 p_2 + \sum_{i=2}^{k-1} \angle p_{i-1} p_i p_{i+1} + \angle p_{k-1} p_k p_1 = (k-2)\pi \qquad \text{(IH)}$$

We now need to show $P(k+1)$ is true. That is, we want to show for *every* convex $(k+1)$-gon, its sum of angles is $(k-1)\pi$. This is a "for all" kind of statement. And to this end, we fix an *arbitrary* $(k+1)$-gon $(p_1, \ldots, p_k, p_{k+1})$ and just prove that this polygon's angles sum to $(k-1)\pi$. Then we would be done by induction, and the remainder of the proof establishes this.

The main idea is this. Connect the line segment $p_1 p_k$. Since $(p_1, \ldots, p_{k+1})$ is convex, it splits this polygon into two convex polygons: the triangle (p_k, p_{k+1}, p_1) and the convex polygon $(p_1, \ldots, p_k)$. Furthermore, two original angles split up thus:

$$\angle p_{k+1} p_1 p_2 = \angle p_{k+1} p_1 p_k + \angle p_k p_1 p_2 \qquad (2)$$

and

$$\angle p_{k-1} p_k p_{k+1} = \angle p_{k-1} p_k p_1 + \angle p_1 p_k p_{k+1} \qquad (3)$$

Refer to Figure 1 again (where you should mentally replace n by $k+1$).

Now, note that the sum of the angles of $(p_1, \ldots, p_{k+1})$, let's call it A, is

$$\begin{aligned} A &= \angle p_{k+1}p_1p_2 + \sum_{i=2}^{k} \angle p_{i-1}p_ip_{i+1} + \angle p_kp_{k+1}p_1 \\ &= \angle p_{k+1}p_1p_2 + \left(\sum_{i=2}^{k-1} \angle p_{i-1}p_ip_{i+1} + \angle p_{k-1}p_kp_{k+1} \right) + \angle p_kp_{k+1}p_1 \\ &= \angle p_{k+1}p_1p_2 + \sum_{i=2}^{k-1} \angle p_{i-1}p_ip_{i+1} + \Big(\angle p_{k-1}p_kp_1 + \angle p_1p_kp_{k+1} \Big) + \angle p_kp_{k+1}p_1 \qquad (4) \\ &= \Big(\angle p_{k+1}p_1p_k + \angle p_kp_1p_2 \Big) + \sum_{i=2}^{k-1} \angle p_{i-1}p_ip_{i+1} + \Big(\angle p_{k-1}p_kp_1 + \angle p_1p_kp_{k+1} \Big) + \angle p_kp_{k+1}p_1 \qquad (5) \end{aligned}$$

where we applied (2) to get (4) and (3) to get (5). Next, we rearrange to get

$$A = \left(\angle p_kp_1p_2 + \sum_{i=2}^{k-1} \angle p_{i-1}p_ip_{i+1} + \angle p_{k-1}p_kp_1 \right) + \Big(p_{k+1}p_1p_k + \angle p_1p_kp_{k+1} + \angle p_kp_{k+1}p_1 \Big)$$

Since $(p_1, \ldots, p_k)$ is some convex polygon on k-vertices, we get that the first parenthesized expression is $(k-2)\pi$ by (IH). This is because (IH) holds for *every* convex polygon on k vertices. The second parenthesized term above adds up to π because it is the sum of angles of the triangle (p_1, p_k, p_{k+1}). Therefore, $A = (k-2)\pi + \pi = (k-1)\pi$, which is what we wanted to show. □

Answers to Exercises.

- **Exercise:** *Using induction, prove $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ for any integer $a > 1$ and non-negative integer n.*

Proof. Fix any real $a > 1$. Let $P(n)$ be the predicate which takes the value true if $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$. We need to prove $\forall n \in \mathbb{N} \cup \{0\} : P(n)$. We proceed by inductions.

Base Case. We need to prove $P(0)$ is true. That is, $\sum_{i=0}^{0} a^i = \frac{a-1}{a-1}$. Indeed, both LHS and RHS are 1.

Inductive Case. Fix $k \geq 0$ and suppose $P(k)$ is true. That is, $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$. We need to prove $P(k+1)$ is true.

Now note,

$$\sum_{i=0}^{k+1} a^i = a^{k+1} + \sum_{i=0}^{k} a^i \underbrace{=}_{P(k)} a^{k+1} + \frac{a^{k+1}-1}{a-1} \underbrace{=}_{\text{algebra}} \frac{\left(a^{k+2} - a^{k+1}\right) + \left(a^{k+1} - 1\right)}{a-1}$$

And now we see that the RHS is $\frac{a^{k+2}-1}{a-1}$, thereby establishing $P(k+1)$. And thus, we have proved the statement by induction. □

- **Exercise:** *Does 4 divide $n^4 - n$ for all non-negative integers n? Mimic the above proof.*

Actually, 4 **does not** divide all $n^4 - n$. Rather than giving you a counterexample, let me actually take you down a "proof", which will **fail** and thus give us a counter example.

"Proof" Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 4 divides $1^4 - 1 = 0$. Indeed, 4 times 0 is 0.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 4 divides $k^4 - k$. We need to show $P(k+1)$ is true, that is, 4 divides $(k+1)^4 - (k+1)$. To do so, we expand $(k+1)^4$, to get

$$(k+1)^4 - (k+1) = (k^4 + 4k^3 + 6k^2 + 4k + 1) - (k+1) = (k^4 - k) + 4(k^3 + k) + 6k^2$$

And now we see our problem. To assert 4 divides $(k+1)^4 - (k+1)$, we see that 4 must divide $6k^2$. This is because 4 does divide $k^4 - k$ (by induction hypothesis) and 4 divides $4(k^3 + k)$. But does 4 divide $6k^2$ always? No! Not when $k = 1$. And so, it suggests for $k = 1$, $P(k+1)$ **may not** be true. That is $P(2)$ may not be true.

Indeed, 4 **does not** divide $2^4 - 2 = 14$. Ta da!

- **Exercise:** *Can you come up with statements like above? Can you guess which number will always divide $4^{3n} - 3^{2n}$ for all natural numbers n? After guessing, can you prove that guess using induction.*

Did you guess? It's $55 = 4^3 - 3^2$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 55 divides $4^3 - 3^2 = 55$. Indeed it does. Therefore $P(1)$ is true.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 55 divides $4^{3k} - 3^{2k}$. We need to show $P(k+1)$ is true, that is, 55 divides $4^{3(k+1)} - 3^{2(k+1)}$. Indeed observe,

$$\begin{aligned} 4^{3(k+1)} - 3^{2(k+1)} &= 4^3 \cdot 4^{3k} - 3^2 \cdot 3^{2k} \\ &= 64 \cdot 4^{3k} - 9 \cdot 3^{2k} \\ &= 64 \cdot 4^{3k} - 64 \cdot 3^{2k} + 64 \cdot 3^{2k} - 9 \cdot 3^{2k} \qquad (6) \\ &= 64 \cdot \left(4^{3k} - 3^{2k}\right) + 55 \cdot 3^{2k} \qquad (7) \end{aligned}$$

55 divides $4^{3k} - 3^{2k}$, by the induction hypothesis. 55 clearly divides $55 \cdot 3^{2k}$. Therefore, 55 divides $4^{3(k+1)} - 3^{2(k+1)}$. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. □