On Column-restricted and Priority Covering Integer Programs

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Abstract

In a 0,1-covering integer program (CIP), the goal is to pick a minimum cost subset of columns of a 0,1-matrix such that for every row, the total number of 1's in the row among the picked columns is at least a specified *demand* of the row. In the *capacitated* version of the problem, each column has an associated *supply*, and in a feasible solution, for every row the total supply in the row among the picked columns is at least the specified demand. The corresponding covering integer program is called the *column-restricted covering integer program* (CCIP) since its constraint matrix has the special property that all non-zeros in any given column are equal to the column's supply.

From an approximation algorithms point of view, CCIPs are not as well understood as their 0,1 counterparts. Our main result connects the approximability of a CCIP with two natural related 0,1-CIPs. The first is the underlying original 0,1-CIP obtained by setting all supplies of columns to 1. The second is a priority version of the 0,1-CIP in which every column and row are assigned priorities. A column now covers a row iff it covers the row in the original 0,1-CIP, and the column's priority exceeds that of the row's. We show that a strengthening of the natural LP relaxation for the CCIP has integrality gap at most $O(\gamma + \omega)$ if the two 0,1-CIPs have gap $O(\gamma)$ and $O(\omega)$, respectively.

Priority versions of CIPs naturally capture *quality of service* type constraints in a covering problem. We study the priority versions of the line (PLC) and the (rooted) tree cover (PTC) problems. Apart from being natural objects to study, these problems fall in a class of fundamental geometric covering problems. We make progress in understanding the integrality gaps of the corresponding PCIPs. Algorithmically, we give a polytime exact algorithm for PLC, show that the PTC problem is APX-hard, and give a factor 2-approximation algorithm for it.

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1 Introduction

In a *0,1-covering integer program* (0,1-CIP, in short), we are given a constraint matrix $A \in \{0,1\}^{m \times n}$, demands $b \in \mathbb{Z}_+^m$, non-negative costs $c \in \mathbb{Z}_+^n$, and upper bounds $d \in \mathbb{Z}_+^n$, and the goal is to solve the following integer linear program (which we denote by $\texttt{Cov}(A, b, c, d)$).

$$\min\{c^T x : Ax \geq b, 0 \leq x \leq d, x \text{ integer}\}.$$

Problems that can be expressed as 0,1-CIPs are essentially equivalent to set multi-cover problems, where sets correspond to columns and elements correspond to rows. This directly implies that 0,1-CIPs are rather well understood in terms of approximability: the class admits efficient $O(\log n)$ approximation algorithms and this is best possible unless NP = P. Nevertheless, in many cases one can get better approximations by exploiting the structure of matrix A. For example, it is well known that whenever A is *totally unimodular* (TU)(see [19] for a definition), the canonical LP relaxation of a 0,1-CIP is integral, and therefore, 0,1-CIPs with this property trivially have polynomial-time exact algorithms.

In this paper, we consider the more general class of covering problems that can be modeled by *column-restricted covering integer programs* (CCIPs). The constraint matrix of a CCIP arises from a 0,1-CIP by multiplying each column j of matrix A with a non-negative *supply* s_j. CCIPs naturally capture *capacitated* versions of 0,1-covering problems. For example, in the capacitated set cover problem, each set has an integer supply of capacity and covers each of its elements that many times. To give one more example, connectivity problems arising in network design applications are often modelled as covering problems. Here, elements correspond to cut sets in an underlying graph which need to be covered by edges or paths – the sets in these applications. As edges and paths correspond to physical links in such applications, supplies naturally model link properties such as capacity or bandwidth.

While a number of general techniques have been developed for obtaining approximation algorithms for structured 0,1-CIPs, not much is known for the column-restricted versions. For instance, it is not known whether constant factor (or, in fact, $o(\log n)$ factor) approximation algorithms exist for CCIPs when the constraint matrix of the underlying 0,1-CIP is totally unimodular.

In this paper, we investigate the relationship between CCIPs and their 0,1 counterparts. In particular, we show that a CCIP has a constant factor approximation if two related 0,1-CIPs have constant integrality gaps. The first is the underlying original 0,1-CIP. The second is a priority version of the 0,1-CIP (PCIPs, in short) in which every column and row are assigned priorities. A column now covers a row iff it covers the row in the original 0,1-CIP, and the column's priority exceeds that of the row's.

Since PCIPs are 0,1 covering integer programs, techniques developed for 0,1-CIPs could be used to bound their integrality gaps; using our result this would imply better algorithms for the CCIPs. Furthermore, PCIPs arise naturally when one wants to implement *quality of service* (QoS) or priority restrictions on a covering problem. These reasons motivate the study of priority versions of covering problems.

In this paper, we study the priority version of the tree covering problem. This is a natural 0,1 covering problem; furthermore, the constraint matrix of this 0,1-CIP is TU, and thus the problem is polynomial time solvable. We show that the problem's priority version, however, is APX-hard, and present constant upper bounds for the integrality gap of this PCIP in a number of special cases. Algorithmically, we show a non-trivial factor 2 approximation for the problem. Besides being a natural covering problem to study, we show that the priority tree cover problem is a special case of a classical geometric covering problem: that of finding a minimum cost cover of points by axis-parallel rectangles in 3 dimensions. Finding a constant factor approximation algorithm for this problem, even when the rectangles have uniform cost, is a long standing open problem. We believe our methods for the priority tree cover problem could give rise to new attacks on the rectangle cover problem.

1.1 Preliminaries

Given a 0,1-CIP $\texttt{Cov}(A, b, c, d)$, we obtain its *canonical LP relaxation* by removing the integrality constraint. The *integrality gap* of the CIP is defined as the supremum of the ratio of optimal IP value to optimal LP value, taken over all demand vectors b, all cost vectors c and all upper-bound vectors d. The integrality gap of an IP captures how much the integrality constraint affects the optimum, and is an indicator of the *strength* of a linear programming formulation.

We now define CCIPs and PCIPs formally. To illustrate these two versions, we use the following two examples of 0,1-covering problems.

Example 1 (Tree and Line Cover)**.** The input is a tree $T = (V, E)$ rooted at a vertex $r \in V$, a set of *segments* $\mathcal{S} \subseteq \{(u, v) : u \text{ is a child of } v\}$, non-negative costs c_j for all $j \in \mathcal{S}$, and demands $\pi_e \in \mathbb{Z}_+$ for all $e \in E$. An edge e is contained in a segment $j = (u, v)$ if e lies on the unique u, v-path in T. The goal is to find a minimum-cost subset C of segments such that each edge $e \in E$ is contained in at least π_e segments of C. When T is just a line, we call the above problem, the *line cover* (LC) problem. In this example, the constraint matrix A has a row for each edge of the tree and a column for each segment in $\mathcal{S}$. It can be shown that A is a TU matrix and thus both these problems can be solved exactly in polynomial time.

Column-Restricted Covering IPs (CCIPs) In the above problem, suppose each segment $j \in \mathcal{S}$ also has a capacity supply s_j associated with it, and call an edge e covered by a collection of segments C iff the total supply of the segments containing e exceeds the demand of e. This is the column-restricted tree cover problem. Observe that if one considers the constraint matrix of this capacitated problem, then in any column corresponding to a segment j each entry is either 0 or s_j.

Given a 0,1-covering problem $\texttt{Cov}(A, b, c, d)$ and a supply vector $s \in \mathbb{Z}^n_+$, the corresponding CCIP is obtained as follows. Let $A[s]$ be the matrix obtained by replacing all the 1's in the jth column by s_j; that is, $A[s]_{ij} = A_{ij} s_j$ for all $1 \le i \le m, 1 \le j \le n$. The column-restricted covering problem is given by the following integer program.

$$\min\{c^T x \,:\, A[s]x \ge b, 0 \le x \le d, x \text{ integer}\}. \tag{$\texttt{Cov}(A[s], b, c, d)$}$$

Priority versions of Covering IPs (PCIPs) In the above problem, suppose each segment j has a *quality of service* (QoS) or priority supply s_j associated with it and suppose each edge e has a QoS or priority demand π_e associated with it. We say that a segment j covers e iff j contains e *and* the priority supply of j exceeds the priority demand of e. The goal is to find a minimum cost subset of segments which covers every edge. This is the priority tree cover problem. We remark that the demand vector for the priority problem is the all 1's vector; one could modify the definition with a more general demand vector, but this definition suffices for our purposes.

In general, the PCIP of a covering problem is obtained as follows. Given a 0,1-covering problem $\texttt{Cov}(A, b, c, d)$, a priority supply vector $s \in \mathbb{Z}^n_+$, and a priority demand vector $\pi \in \mathbb{Z}^m_+$, the corresponding PCIP is as follows. Define $A[s, \pi]$ to be the following 0,1 matrix

$$A[s, \pi]_{ij} = \begin{cases} 1 & : \quad A_{ij} = 1 \text{ and } s_j \ge \pi_i \\ 0 & : \quad \text{otherwise,} \end{cases} \tag{1}$$

Thus, a column j covers row i, only if its priority supply is higher than the priority demand of row i. The priority covering problem is now as follows.

$$\min\{c^T x \,:\, A[s,\pi]x \geq \mathbb{1}, 0 \leq x \leq d, x \text{ integer}\}. \qquad (\texttt{Cov}(A[s,\pi],\mathbb{1},c))$$

Note that we do not need the upper bounds here since no x_i will be greater than 1 in any minimal solution. We define the integrality gap of PCIP as the supremum, taken over all choices of s, π and c, of the ratio of the optimum value of $\texttt{Cov}(A[s,\pi],\mathbb{1},c)$ to its canonical LP relaxation.

1.2 Related work

There is a rich and long line of work ([10, 12, 18, 20, 21]) on approximation algorithms for CIPs, of which we state the most relevant to our work. Let α, called the *dilation* of a CIP, denote the maximum number of non-zeros in any column. Assuming no upper bounds on the variables, Srinivasan [20] gave a $O(1 + \log \alpha)$-approximation to the problem. Later on, Kolliopoulos and Young [16] obtained the same approximation factor, respecting the upper bounds. However, these algorithms didn't give any better results when special structure of the constraint matrix was known. On the hardness side, Trevisan [22] showed that it is NP-hard to obtain a $(\log \alpha - O(\log\log \alpha))$-approximation algorithm even for 0,1-CIPs.

The most relevant work to this paper is that of Kolliopoulos [13]. The author shows that for CCIPs, if one is allowed to violate the upper bounds by a multiplicative constant, then the integrality gap of the CCIP is within a constant factor of that of the original 0,1-CIP[1]. As the author notes such a violation is necessary; otherwise the CCIP has unbounded integrality gap. If one is not allowed to violated upper bounds, nothing better than the result of [16] is known for the special case of CCIPs. Furthermore, even for the result violating the upper bounds, [13] makes a rather strong assumption, called the *no bottleneck assumption*, of the supply of any column being smaller than the demand of any row. We show that a slightly modified analysis gives a slightly better factor and bases on a slightly weaker assumption.

Our work on CCIPs parallels a large body of work on column-restricted *packing* integer programs (CPIPs). Assuming the *no-bottleneck assumption*, Kolliopoulos and Stein [15] show that CPIPs can be approximated asymptotically as well as the corresponding 0,1-PIPs. Chekuri et al. [7] subsequently improve the constants in the result from [15]. These results imply constant factor approximations for the column-restricted tree *packing* problem under the no-bottleneck assumption. Without the no-bottleneck assumption, however, only polylogarithmic approximation is known for the problem [6].

The only work on priority versions of covering problems that we are aware of is due to Charikar, Naor and Schieber [5] who studied the priority Steiner tree and forest problems in the context of QoS management in a network multicasting application. Charikar et al. present a $O(\log n)$-approximation algorithm for the problem, and Chuzhoy et al. [9] later show that no efficient $o(\log\log n)$ approximation algorithm can exist unless NP $\subseteq$ DTIME$(n^{\log\log\log n})$ (n is the number of vertices).

To the best of our knowledge, the column-restricted or priority versions of the line and tree cover problem have not been studied. The best known approximation algorithm known for both is the $O(\log n)$ factor implied by the results of [16] stated above. However, upon completion of our work, Nitish Korula [17] pointed out to us that a 4-approximation for column-restricted line cover is implicit in a result of Bar-Noy et al. [2]. We remark that their algorithm is not LP-based, although our general result on CCIPs is.

[1]Such a result is implicit in the paper; the author only states a $O(\log \alpha)$ integrality gap.

1.3 Technical Contributions and Formal Statement of Results

CCIPs Suppose the CCIP is $\texttt{Cov}(A[s], b, c, d)$. We make the following two assumptions about the integrality gaps of the 0,1 covering programs, both the original 0,1-CIP and the priority version of the 0,1-CIP.

Assumption 1. *The integrality gap of the original 0,1-CIP is $\gamma \geq 1$. Specifically, for any non-negative integral vectors $b \in \mathbb{Z}_+^m, c \in \mathbb{Z}_+^n$, and $d \in \mathbb{Z}_+^m$, if the canonical LP relaxation to the CIP has a fractional solution x, then one can find in polynomial time an integral feasible solution to the CIP of cost at most $\gamma \cdot c^T x$. We stress here that the entries of b, c, d could be 0 as well as ∞.*

Assumption 2. *The integrality gap of the PCIP is $\omega \geq 1$. Specifically, for any non-negative integral vectors s, π, c, if the canonical LP relaxation to the PCIP has a fractional solution x, then one can find in polynomial time, an integral feasible solution to the PCIP of cost at most $\omega \cdot c^T x$.*

We give an LP-based approximation algorithm for solving CCIPs. Since the canonical LP relaxation of a CCIP can have unbounded integrality gap, we strengthen it by adding a set of valid constraints called the *knapsack cover constraints*. We show that the integrality gap of this strengthened LP is $O(\gamma + \omega)$, and can be used to give a polynomial time approximation algorithm.

Theorem 1. *Under Assumptions 1 and 2, there is a $(24\gamma+8\omega)$-approximation algorithm for column-restricted CIPs.*

Knapsack cover constraints to strengthen LP relaxations were introduced in [1, 11, 23]; Carr et al. [4] were the first to use it in the design approximation algorithms. The paper of Kolliopoulos and Young [16] also use these to get their result on general CIPs.

The main technique used for designing algorithms for column-restricted problems is *grouping-and-scaling* developed by Kolliopoulos and Stein [14, 15] for packing problems, and later used by Kolliopoulos [13] in the covering context. In this technique, the *columns* of the matrix are divided into groups of 'close-by' supply values; in a single group, the supply values are then scaled to be the same; for a single group, the integrality gap of the original 0,1-CIP is invoked to get an integral solution for that group; the final solution is a 'union' of the solutions over all groups.

There are two issues in applying the technique to the new strengthened LP relaxation of our problem. Firstly, although the original constraint matrix is column-restricted, the new constraint matrix with the knapsack cover constraints is not. Secondly, unless additional assumptions are made, the current grouping-and-scaling analysis doesn't give a handle on the degree of violation of the upper bound constraints. This is the reason why Kolliopoulos [13] needs the strong no-bottleneck assumption.

We get around the first difficulty by grouping the *rows* as well, into those which get most of their coverage from columns not affected by the knapsack constraints, and the remainder. On the first group of rows, we apply a subtle modification to the vanilla grouping-and-scaling analysis and obtain a $O(\gamma)$ approixmatefeasible solution satisfying these rows; we then show that one can treat the remainder of the rows as a PCIP and get a $O(\omega)$ approximate feasible solution satisfying them, using Assumption 2. Combining the two gives the $O(\gamma + \omega)$ factor. The full details are given in Section 2.

We stress here that apart from the integrality gap assumptions on the 0,1-CIPs, we do not make any other assumption (like the no-bottleneck assumption). In fact, we can use the modified analysis of the grouping-and-scaling technique to get a similar result as [13] for approximating CCIPs violating the upper-bound constraints, under a *weaker* assumption than the no-bottleneck assumption. The no-bottleneck assumption states that the supply of *any* column is less than the demand of *any* row. In particular, even though a column has entry 0 on a certain row, its supply needs to be less than the demand of that row. We show that if we weaken the no-bottleneck assumption to assuming that the supply of a column j is less than the demand of any row i only if $A[s]_{ij}$ is positive, a similar result can be obtained via our modified analysis. Our constant 10 in the following theorem is slightly smaller than the constant 12 in [13].

Theorem 2. *Under assumption 1 and assuming $A_{ij}s_j \leq b_i$, for all i, j, given a fractional solution x to the canonical LP relaxation of $\texttt{Cov}(A[s], b, c, d)$, one can find an integral solution $x^{\texttt{int}}$ whose cost $c \cdot x^{\texttt{int}} \leq 10\gamma(c \cdot x)$ and $x^{\texttt{int}} \leq 10d$.*

Priority Covering Problems In the following, we use PLC and PTC to refer to the priority versions of the line cover and tree cover problems, respectively. Recall that the constraint matrices for line and tree cover problems are totally unimodular, and the integrality of the corresponding 0,1-covering problems is therefore 1 in both case. It is interesting to note that the 0,1-coefficient matrices for PLC and PTC are not totally unimodular in general. The following integrality gap bound is obtained via a primal-dual algorithm.

Theorem 3. *The canonical LP for priority line cover has an integrality gap of at least* $3/2$ *and at most* 2.

In the case of tree cover, we obtain constant upper bounds on the integrality gap for the case $c = \mathbb{1}$, that is, for the minimum cardinality version of the problem. We believe that the PCIP for the tree cover problem with general costs also has a constant integrality gap. On the negative side, we can show an integrality gap of at least $\frac{e}{e-1}$.

Theorem 4. *The canonical LP for* unweighted *PTC has an integrality gap of at most* 6.

We obtain the upper bound by taking a given PTC instance and a fractional solution to its canonical LP, and decomposing it into a collection of PLC instances with corresponding fractional solutions, with the following two properties. First, the total cost of the fractional solutions of the PLC instances is within a constant of the cost of the fractional solution of the PTC instance. Second, union of integral solutions to the PLC instances gives an integral solution to the PTC instance. The upper bound follows from Theorem 3. Using Theorem 1, we get the following as an immediate corollary.

Corollary 1. *There are $O(1)$-approximation algorithms for column-restricted line cover and the cardinality version of the column-restricted tree cover.*

We also obtain the following combinatorial results.

Theorem 5. *There is a polynomial-time exact algorithm for PLC.*

Theorem 6. *PTC is APX-hard, even when all the costs are unit.*

Theorem 7. *There is an efficient* 2*-approximation algorithm for PTC.*

The algorithm for PLC is a non-trivial dynamic programming approach that makes use of various structural observations about the optimal solution. The approximation algorithm for PTC is obtained via a similar decomposition used to prove Theorem 4.

We end by noting some interesting connections between the priority tree covering problem and set covering problems in computational geometry. The *rectangle cover* problem in 3-dimensions is the following: given a collection of points P in $\mathbb{R}^3$, and a collection C of axis-parallel rectangles with costs, find a minimum cost collection of rectangles which covers every point. Finding a constant factor approximation algorithm is a long standing open problem in computational geometry, even when all costs are unit. We can show the following theorem. We believe, studying the PTC problem could give new insights into the rectangle cover problem.

Theorem 8. *The priority tree covering problem is a special case of the rectangle cover problem in* 3-*dimensions.*

2 General Framework for Column Restricted CIPs

In this section we prove Theorem 1. Our goal is to round a solution to a LP relaxation of $\texttt{Cov}(A[s], b, c, d)$ into an approximate integral solution. We strengthen the following canonical LP relaxation of the CCIP

$$\min\{c^T x \ : \ A[s]x \geq b, 0 \leq x \leq d, x \geq 0\}$$

by adding valid *knapsack cover* constraints. In the following we use $\mathcal{C}$ for the set of columns and $\mathcal{R}$ for the set of rows of A.

2.1 Strengthening the canonical LP Relaxation

Let $F \subset \mathcal{C}$ be a subset of the columns in the column restricted CIP $\texttt{Cov}(A[s], b, c, d)$. For all rows $i \in \mathcal{R}$, define $b_i^F = \max\{0, b_i - \sum_{j \in F} A[s]_{ij} d_j\}$ to be the residual demand of row i w.r.t. F. Define matrix $A^F[s]$ by letting

$$A^F[s]_{ij} = \begin{cases} \min\{A[s]_{ij}, b_i^F\} & : \ j \in \mathcal{C} \setminus F \\ 0 & : \ j \in F, \end{cases} \tag{2}$$

for all $i \in \mathcal{C}$ and for all $j \in \mathcal{R}$. The following *Knapsack-Cover* (KC) inequality

$$\sum_{j \in \mathcal{C}} A^F[s]_{ij} x_j \geq b_i^F$$

is valid for the set of all integer solutions x for $\texttt{Cov}(A[s], b, c, d)$. Adding the set of all KC inequalities yields the following stronger LP formulation CIP. We note that the LP is not column-restricted, in that, different values appear on the same column of the new constraint matrix.

$$\begin{aligned} \texttt{opt}_P := \min \quad & \sum_{j \in \mathcal{C}} c_j x_j && && \text{(P)} \\ \text{s.t.} \quad & \sum_{j \in \mathcal{C}} A^F[s]_{ij} x_j \geq b_i^F && \forall F \subseteq \mathcal{C}, \forall i \in \mathcal{R} && (3) \\ & 0 \leq x_j \leq d_j && \forall j \in \mathcal{C} && \end{aligned}$$

It is not known whether (P) can be solved in polynomial time. For $\alpha \in (0,1)$, call a vector x^* α-relaxed if its cost is at most $\texttt{opt}_P$, and if it satisfies (3) for $F = \{j \in \mathcal{C} \ : \ x_j^* \geq \alpha d_j\}$. An α-relaxed solution to (P) can be computed efficiently for any α. To see this note that one can check whether a candidate solution satisfies (3) for a set F. If a solution satisfies, we are done, otherwise we have found a inequality of (P) which the solution doesn't satisfy and we can make progress via the ellipsoid method. Details can be found in [4] and [16].

We fix an $\alpha \in (0,1)$, specifying its precise value later. Compute an α-relaxed solution, x^*, for (P), and let $F = \{j \in \mathcal{C} : x^*_j \geq \alpha d_j\}$. Define $\bar{x}$ as, $\bar{x}_j = x^*_j$ if $j \in \mathcal{C} \setminus F$, and $\bar{x}_j = 0$, otherwise. Since x^* is an α-relaxed solution, we get that $\bar{x}$ is a feasible fractional solution to the *residual* CIP, $\texttt{Cov}(A^F[s], b^F, c, \alpha d)$. In the next subsection, our goal will be to obtain an *integral* feasible solution to the covering problem $\texttt{Cov}(A^F[s], b^F, c, d)$ using $\bar{x}$. The next lemma shows how this implies an approximation to our original CIP.

Lemma 1. *If there exists an integral feasible solution, $x^{\texttt{int}}$, to $\texttt{Cov}(A^F[s], b^F, c, d)$ with $c^T x^{\texttt{int}} \leq \beta \cdot c^T \bar{x}$, then there exists a $\max\{1/\alpha, \beta\}$-factor approximation to $\texttt{Cov}(A[s], b, c, d)$.*

Proof. Define

$$z_j = \begin{cases} d_j & : \quad j \in F \\ x^{\texttt{int}}_j & : \quad j \in \mathcal{C} \setminus F, \end{cases} \tag{4}$$

Observe that $z \leq d$. z is a feasible integral solution to $\texttt{Cov}(A[s], b, c, d)$ since for any $i \in \mathcal{R}$,

$$\sum_{j \in \mathcal{C}} A[s]_{ij} z_j = \sum_{j \in F} A[s]_{ij} d_j + \sum_{j \in \mathcal{C} \setminus F} A[s]_{ij} x^{\texttt{int}}_j \geq (b_i - b^F_i) + \sum_{j \in \mathcal{C} \setminus F} A^F[s]_{ij} x^{\texttt{int}}_j \geq b_i$$

where the first inequality follows from the definition of b^F_i and since $A[s]_{ij} \geq A^F[s]_{ij}$, the second inequality follows since $x^{\texttt{int}}$ is a feasible solution to $\texttt{Cov}(A^F[s], b^F, c, d)$.
Furthermore,

$$c^T z = \sum_{j \in F} c_j d_j + \sum_{j \in \mathcal{C} \setminus F} c_j x^{\texttt{int}}_j \leq \frac{1}{\alpha} \sum_{j \in F} c_j x^*_j + \beta \sum_{j \in \mathcal{C} \setminus F} c_j x^*_j \leq \max\{\frac{1}{\alpha}, \beta\} \texttt{opt}_P$$

where the first inequality follows from the definition of F and the second from the assumption in the theorem statement. □

2.2 Solving the Residual Problem

In this section we use a feasible fractional solution $\bar{x}$ of $\texttt{Cov}(A^F[s], b^F, c, \alpha d)$, to obtain an *integral* feasible solution $x^{\texttt{int}}$ to the covering problem $\texttt{Cov}(A^F[s], b^F, c, d)$, with $c^T x^{\texttt{int}} \leq \beta c^T \bar{x}$ for $\beta = 24\gamma + 8\omega$. Fix $\alpha = 1/24$.

***Converting to Powers of* 2.** For ease of exposition, we first modify the input to the residual problem $\texttt{Cov}(A^F[s], b^F, c, d)$ so that all entries of are powers of 2. For every $i \in \mathcal{R}$, let $\bar{b}_i$ denote the smallest power of 2 larger than b^F_i. For every column $j \in \mathcal{C}$, let $\bar{s}_j$ denote the largest power of 2 smaller than s_j.

Lemma 2. *$y = 4\bar{x}$ is feasible for $\texttt{Cov}(A^F[\bar{s}], \bar{b}, c, 4\alpha d)$.*

Proof. Focus on row $i \in \mathcal{R}$. We have

$$\sum_{j \in \mathcal{C}} A^F[\bar{s}]_{ij} y_j \geq 2 \cdot \sum_{j \in \mathcal{C}} A^F[s]_{ij} \bar{x}_j \geq 2b^F_i \geq \bar{b}_i,$$

where the first inequality uses the fact that $s_j \leq 2\bar{s}_j$ for all $j \in \mathcal{C}$, the second inequality uses the fact that $\bar{x}$ is feasible for $\texttt{Cov}(A^F[s], b^F, c, \alpha d)$, and the third follows from the definition of $\bar{b}_i$. □

Partitioning the rows. We call $\bar{b}_i$ the residual demand of row i. For a row i, a column $j \in \mathcal{C}$ is *i-large* if the supply of j is at least the residual demand of row i; it is *i-small* otherwise. Formally,

$$\begin{aligned}
\mathcal{L}_i &= \{j \in \mathcal{C} : A_{ij} = 1, \bar{s}_j \geq \bar{b}_i\} \quad \text{is the set of } i\text{-large columns} \\
\mathcal{S}_i &= \{j \in \mathcal{C} : A_{ij} = 1, \bar{s}_j < \bar{b}_i\} \quad \text{is the set of } i\text{-small columns}
\end{aligned}$$

Recall the definition from (2), $A^F[\bar{s}]_{ij} = \min(A[\bar{s}]_{ij}, b_i^F)$. Therefore, $A^F[\bar{s}]_{ij} = A_{ij}b_i^F$ for all $j \in \mathcal{L}_i$ since $\bar{s}_j \geq \bar{b}_i \geq b_i^F$; and $A^F[\bar{s}]_{ij} = A_{ij}\bar{s}_j$ for all $j \in \mathcal{S}_i$, since being powers of 2, $\bar{s}_j < \bar{b}_i$ implies, $\bar{s}_j \leq \bar{b}_i/2 \leq b_i^F$.

We now partition the rows into large and small depending on which columns most of their coverage comes from. Formally, call a row $i \in \mathcal{R}$ *large* if

$$\sum_{j \in \mathcal{S}_i} A^F[\bar{s}]_{ij}y_j \leq \sum_{j \in \mathcal{L}_i} A^F[\bar{s}]_{ij}y_j,$$

and small otherwise. Note that Lemma 2 together with the fact that each column in row i's support is either small or large implies,

$$\text{For a large row } i, \ \sum_{j \in \mathcal{L}_i} A^F[\bar{s}]_{ij}y_j \geq \bar{b}_i/2, \qquad \text{For a small row } i, \ \sum_{j \in \mathcal{S}_i} A^F[\bar{s}]_{ij}y_j \geq \bar{b}_i/2$$

Let $\mathcal{R}_L$ and $\mathcal{R}_S$ be the set of large and small rows.

In the following, we address small and large rows separately. We compute a pair of integral solutions $x^{\texttt{int},\mathcal{S}}$ and $x^{\texttt{int},\mathcal{L}}$ which are feasible for the small and large rows, respectively. We then obtain $x^{\texttt{int}}$ by letting

$$x_j^{\texttt{int}} = \max\{x_j^{\texttt{int},\mathcal{S}}, x_j^{\texttt{int},\mathcal{L}}\}, \tag{5}$$

for all $j \in \mathcal{C}$.

2.2.1 Small rows.

For these rows we use the grouping-and-scaling technique a la [14, 15, 7, 13]. However, as mentioned in the introduction, we use a modified analysis which bypasses the no-bottleneck assumptions made by earlier works.

Lemma 3. *We can find an integral solution $x^{\texttt{int},\mathcal{S}}$ such that*
a) $x_j^{\texttt{int},\mathcal{S}} \leq d_j$ for all j,
b) $\sum_{j \in \mathcal{C}} c_j x_j^{\texttt{int},\mathcal{S}} \leq 24\gamma \sum_{j \in \mathcal{C}} c_j \bar{x}_j$, and
c) for every small row $i \in \mathcal{R}_S$, $\sum_{j \in \mathcal{C}} A^F[s]_{ij} x_j^{\texttt{int},\mathcal{S}} \geq b_i^F$.

Proof. The complete proof is slightly technical and hence we start with a sketch. Since the rows are small, for any row i, we can zero out the entries which are larger than $\bar{b}_i$, and still $2y$ will be a feasible solution. Note that, now in each row, the entries are $< \bar{b}_i$, and thus are at most $\bar{b}_i/2$ (everything being powers of 2). We stress that it could be that $\bar{b}_i$ of some row is less than the entry in some other row, that is, we don't have the no-bottleneck assumption. However, we have the weaker assumption and our modified analysis of grouping and scaling will make the proof go through.

We *group* the columns into classes which have s_j as the same power of 2, and for each row i we let $\bar{b}_i^{(t)}$ be the contribution of the class t columns towards the demand of row i. The columns of class t, the small rows, and the demands $\bar{b}_i^{(t)}$ form a CIP where all non-zero entries of the matrix are the same power of 2. We scale both the constraint matrix and $\bar{b}_i^{(t)}$ down by that power of 2 to get a 0,1-CIP, and using assumption 1, we get an integral solution to this 0,1-CIP. Our final integral solution is obtained by concatenating all these integral solutions over all classes.

Till now the algorithm is the standard grouping-and-scaling algorithm. The difference lies in our analysis in proving that this integral solution is feasible for the original CCIP. Originally the no-bottleneck assumption was used to make this go through. However, we show since the column values in different classes are geometrically decreasing, the weaker assumption can be made to go through. We have skipped the details, and in fact an extra scaling is necessary to make the argument go through. We now get into the full proof.

Step 1: Grouping the columns:

Let $\bar{s}_{min}$ and $\bar{s}_{max}$ be the smallest and largest supply among the columns in $\mathcal{C} \setminus F$. Since all $\bar{s}_j$ are powers of 2, we introduce the shorthand, $\bar{s}^{(t)}$ for the supply $\bar{s}_{\max}/2^t$. We say that a column j is in *class* $t \geq 0$, if $\bar{s}_j = \bar{s}^{(t)}$, and we let

$$\mathcal{C}^{(t)} := \{j \in \mathcal{C} \setminus F \,:\, \bar{s}_j = \bar{s}^{(t)}\}$$

be the set of class t supplies.

Step 2: Disregarding i-large columns of a small row i

Fix a small row $i \in \mathcal{R}_S$. We now identify the columns j which are i-small. To do so, define $t_i := \log(\bar{s}_{max}/\bar{b}_i) + 1$. Observe that any column j in class $\mathcal{C}^{(t)}$ for $t \geq t_i$ are i-small. This is because $\bar{s}_j = s_{max}/2^t \leq s_{max}/2^{t_i} = \bar{b}_i/2 < \bar{b}_i$. Define

$$\bar{b}_i^{(t)} = \begin{cases} 2\sum_{j \in \mathcal{C}^{(t)}} A^F[\bar{s}]_{ij} y_j & : \quad t \geq t_i \\ 0 & : \quad \text{otherwise} \end{cases}$$

as the contribution of the class t, i-small columns to the demand of row i, multiplied by 2. Note that by definition of small rows, these columns contribute to more than 1/2 of the demand, and so

$$\sum_{t \geq t_i} \bar{b}_i^{(t)} \geq \bar{b}_i. \tag{6}$$

Henceforth, we will consider only the contributions of the small i-columns of a small row i.

Step 3: Scaling and getting the integral solution

Fix a class t of columns and scale down by $\bar{s}^{(t)}$ to get a $\{0,1\}$-constraint matrix. (Recall entries of the columns in a class t are all $\bar{s}^{(t)}$.) This will enable us to apply assumption 1 and get a integral solution corresponding to these columns. The final integral solution will be the concatenation of the integral solutions over the various classes.

The constants in the next claim are carefully chosen for the calculations to work out later.

Claim 1. *For any $t \geq 0$ and for all $i \in \mathcal{R}_S$, $6 \cdot \sum_{j \in \mathcal{C}^{(t)}} A_{ij} y_j \geq \lfloor 3\bar{b}_i^{(t)}/\bar{s}^{(t)} \rfloor$.*

Proof. The claim is trivially true for rows i with $t_i > t$ as $\bar{b}_i^{(t)} = 0$ in this case. Consider a row i with $t_i \leq t$. Since any column $j \in \mathcal{C}^{(t)}$ is i-small, we get $A^F[\bar{s}]_{ij} = A_{ij}\bar{s}_j = A_{ij}\bar{s}^{(t)}$. Using the definition of $\bar{b}_i$, we obtain

$$6 \cdot \sum_{j \in \mathcal{C}^{(t)}} A_{ij}\bar{s}^{(t)} y_j = 3\bar{b}_i^{(t)}.$$

Dividing both sides by $\bar{s}^{(t)}$ and taking the floor on the right-hand side yields the claim. □

Since $\alpha = 1/24$ and $\bar{x}$ is a feasible solution to $\texttt{Cov}(A^F[s], b^F, c, d/24)$, we get that $6y_j = 24 \cdot \bar{x}_j \leq d_j$ for all $j \in \mathcal{C} \setminus F$. Thus, the above claim shows that $6y$ is a feasible fractional solution for $\texttt{Cov}(A^{(t)}, \lfloor 3\bar{b}^{(t)}/\bar{s}^{(t)} \rfloor, c^{(t)}, d^{(t)})$, where $A^{(t)}$ is the submatrix of A defined by the columns in $\mathcal{C}^{(t)}$, and $c^{(t)}$ and $d^{(t)}$ are the sub-vectors of c and d, respectively, that are induced by $\mathcal{C}^{(t)}$. Using Assumption 1, we therefore conclude that there is an integral vector $x^{\texttt{int},\mathcal{S},t}$ such that

$$x_j^{\texttt{int},\mathcal{S},t} \leq d_j \quad \text{for all } j \in \mathcal{C}^{(t)}, \text{ and} \tag{7}$$

$$\sum_{j \in \mathcal{C}^{(t)}} A_{ij}^{(t)} x_j^{\texttt{int},\mathcal{S},t} \geq \left\lfloor \frac{3\bar{b}_i^{(t)}}{\bar{s}^{(t)}} \right\rfloor \quad \text{for all } i \in \mathcal{R}_S, \text{ and} \tag{8}$$

$$\sum_{j \in \mathcal{C}^{(t)}} c_j x_j^{\texttt{int},\mathcal{S},t} \leq 6\gamma \cdot \sum_{j \in \mathcal{C}^{(t)}} c_j y_j \tag{9}$$

We obtain integral solution $x^{\texttt{int},\mathcal{S}}$ by letting $x_j^{\texttt{int},\mathcal{S}} = x_j^{\texttt{int},\mathcal{S},t}$ if $j \in \mathcal{C}^{(t)}$. Thus $x_j^{\texttt{int},\mathcal{S}} \leq d_j$ for all $j \in \mathcal{C}$, and we get,

$$\sum_{j \in \mathcal{C}} c_j x_j^{\texttt{int},\mathcal{S}} = \sum_{t \geq 0} \sum_{j \in \mathcal{C}^{(t)}} c_j x_j^{\texttt{int},\mathcal{S},t} \leq 6\gamma \cdot \sum_{t \geq 0} \sum_{j \in \mathcal{C}^{(t)}} c_j y_j = 24\gamma \cdot \sum_{j \in \mathcal{C}} c_j \bar{x}_j. \tag{10}$$

Thus we have established parts (a) and (b) of the lemma. It remains to show that $x^{\texttt{int},\mathcal{S}}$ is feasible for the set of small rows.

Step 4: Putting them all together: scaling back

Once again, fix a small row $i \in \mathcal{R}_S$. The following inequality takes only contribution of the i-small columns. We later show this suffices.

$$\begin{aligned} \sum_{j \in \mathcal{C}} A^F[s]_{ij} x_j^{\texttt{int},\mathcal{S}} &\geq \sum_{j \in \mathcal{C}:\ j \text{ is } i\text{-small}} A_{ij} s_j x_j^{\texttt{int},\mathcal{S}} \\ &= \sum_{t \geq t_i} \sum_{j \in \mathcal{C}^{(t)}} A_{ij}^{(t)} s_j x_j^{\texttt{int},\mathcal{S}} = \sum_{t \geq t_i} \sum_{j \in \mathcal{C}^{(t)}} A_{ij}^{(t)} \bar{s}^{(t)} x_j^{\texttt{int},\mathcal{S},t} \end{aligned} \tag{11}$$

The first inequality follows since $A^F[s]_{ij} = A_{ij} s_j$ for i-small columns, and the second equality follows from the definition of t_i and $\bar{s}^{(t)}$. The following claim along with (11) proves feasibility of row i. This is the part where our analysis slightly differs from the standard grouping-and-scaling analysis.

Claim 2. *For any small row $i \in \mathcal{R}_S$,*

$$\sum_{t \geq t_i} \sum_{j \in \mathcal{C}^{(t)}} A_{ij}^{(t)} \bar{s}^{(t)} x_j^{\texttt{int},\mathcal{S},t} \geq b_i^F.$$

Proof. In this proof, the choice of the constant 3 will become clear. Let $S_i = \{t \geq t_i : 3\bar{b}_i^{(t)} < \bar{s}^{(t)}\}$ be the set of i-small classes whose supply towards row i is relatively small. We now show that for any small row i, the columns in the classes not in S_i suffice to satisfy its demand. Note that

$$\sum_{t \notin S_i, t \geq t_i} \bar{b}_i^{(t)} = \sum_{t \geq t_i} \bar{b}_i^{(t)} - \sum_{t \in S_i} \bar{b}_i^{(t)} \geq \sum_{t \geq t_i} \bar{b}_i^{(t)} - \frac{1}{3} \sum_{t \in S_i} \bar{s}^{(t)} \tag{12}$$

which follows from the definition of S_i. Furthermore, from (6) we know that for a small row, $\sum_{t\geq t_i} \bar{b}_i^{(t)} \geq \bar{b}_i$. Also, since $\bar{s}^{(t)}$ form a geometric series, we get that $\sum_{t\in S_i} \bar{s}^{(t)} \leq \sum_{t\geq t_i} \bar{s}^{(t)} \leq 2\bar{s}^{(t_i)}$. Putting this in (12) we get

$$\sum_{t\notin S_i, t\geq t_i} \bar{b}_i^{(t)} \geq \bar{b}_i - \frac{1}{3}\sum_{t\geq t_i} \bar{s}^{(t)} \geq \bar{b}_i - \frac{2}{3}\bar{s}^{(t_i)} = \frac{2}{3}\bar{b}_i, \tag{13}$$

where the final equality follows from the definition of t_i which implies that $\bar{s}^{(t_i)} = \bar{b}_i/2$.

Moreover, for $t \notin S_i$, we know that $\lfloor 3\bar{b}_i^t/\bar{s}^{(t)} \rfloor \geq \frac{3}{2}\bar{b}_i^t/\bar{s}^{(t)}$ since $\lfloor a \rfloor \geq a/2$ if $a > 1$. Therefore, using inequality (8) in (11), we get

$$\begin{aligned}
\sum_{j\in\mathcal{C}} A^F[s]_{ij} x_j^{\texttt{int},\mathcal{S}} \geq \sum_{t\geq t_i}\sum_{j\in\mathcal{C}^{(t)}} A_{ij}^{(t)} \bar{s}^{(t)} x_j^{\texttt{int},\mathcal{S},t} &\geq \sum_{t\notin S_i, t\geq t_i} \bar{s}^{(t)} \left\lfloor \frac{3\bar{b}_i^{(t)}}{\bar{s}^{(t)}} \right\rfloor \\
&\geq \frac{3}{2}\sum_{t\notin S_i, t\geq t_i} \bar{b}_i^{(t)} \\
&\geq \bar{b}_i \geq b_i^F,
\end{aligned}$$

where the second-last inequality uses (13), and the last uses the definition of $\bar{b}_i$. This completes the proof of the lemma. □

□

2.2.2 Large rows.

The large rows can be showed to be a PCIP problem and thus Assumption 2 can be invoked to get an analogous lemma to Lemma 3.

Lemma 4. *We can find an integral solution $x^{\texttt{int},\mathcal{L}}$ such that*
a) $x_j^{\texttt{int},\mathcal{L}} \leq 1$ for all j,
b) $\sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int},\mathcal{S}} \leq 8\omega \sum_{j\in\mathcal{C}} c_j \bar{x}_j$, and
c) for every large row $i \in \mathcal{R}_L$, $\sum_{j\in\mathcal{C}} A^F[s]_{ij} x_j^{\texttt{int},\mathcal{S}} \geq b_i^F$.

Proof. Let $i \in \mathcal{R}_L$ be a large row, and recall that $\mathcal{L}_i$ is the set of i-large columns in $\mathcal{C}$. We have

$$\sum_{j\in\mathcal{L}_i} A^F[s]_{ij} y_j = \sum_{j\in\mathcal{L}_i} A_{ij}\bar{b}_i y_j \geq \bar{b}_i/2,$$

and hence

$$2\sum_{j\in\mathcal{L}_i} A_{ij} y_j \geq 1. \tag{14}$$

Let $A^{\mathcal{R}}$ be the minor of A induced by the large rows. Consider the priority cover problem $\texttt{Cov}(A^{\mathcal{R}}[\bar{s},\bar{b}], \mathbb{1}, c)$. From the definition of $\mathcal{L}_i$, it follows $2y$ is a feasible fractional solution to the priority cover problem.

Using Assumption 2, we conclude that there is an integral solution $x^{\texttt{int},\mathcal{L}}$ such that $\sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int},\mathcal{L}} \leq 2\omega \sum_{j\in\mathcal{C}} c_j y_j = 8\omega \sum_{j\in\mathcal{C}} c_j \bar{x}_j$, and $\sum_{j\in\mathcal{C}} A_{ij}^{\mathcal{R}} x_j^{\texttt{int},\mathcal{L}} \geq 1$, for all large rows $i \in \mathcal{R}_L$.

Fix a large row i. Since $A^F[s]_{ij} = b_i^F$ for all i-large columns $\mathcal{L}_i$, we get

$$\sum_{j\in\mathcal{C}} A^F[s]_{ij} x_j^{\texttt{int},\mathcal{L}} \geq \sum_{j\in\mathcal{L}_i} A_{ij} b_i^F x_j^{\texttt{int},\mathcal{L}} = b_i^F \sum_{j\in C} A_{ij}^{\mathcal{R}} x_j^{\texttt{int},\mathcal{L}} \geq b_i^F$$

This completes the proof of the lemma. □

Proof of Theorem 1 Let $x^{\texttt{int},\mathcal{S}}$ and $x^{\texttt{int},\mathcal{L}}$ be as satisfying the conditions of Lemma 3 and 4, respectively. Define $x^{\texttt{int}}$ as $x_j^{\texttt{int}} = \max\{x_j^{\texttt{int},\mathcal{S}}, x_j^{\texttt{int},\mathcal{L}}\}$. We have

a) $x_j^{\texttt{int}} \le d_j$ since both $x_j^{\texttt{int},\mathcal{S}} \le d_j$ and $x_j^{\texttt{int},\mathcal{L}} \le 1 \le d_j$.

b) For any row i, $\sum_{j\in\mathcal{C}} A^F[s]_{ij}x_j^{\texttt{int}} \ge b_i^F$ since the inequality is true with $x^{\texttt{int}}$ replaced by $x^{\texttt{int},\mathcal{S}}$ for small rows, and $x^{\texttt{int}}$ by $x^{\texttt{int},\mathcal{L}}$ for large rows.

c) $\sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int}} \le \sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int},\mathcal{S}} + \sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int},\mathcal{L}} \le (24\gamma + 8\omega)\sum_{j\in\mathcal{C}} c_j\bar{x}_j$.

Thus, $x^{\texttt{int}}$ is a feasible integral solution to $\texttt{Cov}(A^F[s], b^F, c, d)$ with cost bounded as $\sum_{j\in\mathcal{C}} c_j x_j^{\texttt{int}} \le (24\gamma + 8\omega)\sum_{j\in\mathcal{C}} c_j\bar{x}_j$. Noting that $\alpha = 1/24$, the proof of the theorem follows from Lemma 1. $\square$.

2.3 CCIPs with violation of upper-bounds: Proof of Theorem 2

In this section we prove Theorem 2 which we restate here. In the proof, we will indicate how we modify the analysis of grouping-and-scaling which allows us to replace the no-bottleneck assumption with a weaker one.

Theorem 9. *(Theorem 2) Under assumption 1 and assuming $A_{ij}s_j \le b_i$, for all i, j, given a fractional solution x to the canonical LP relaxation of* $\texttt{Cov}(A[s], b, c, d)$*, one can find an integral solution $x^{\texttt{int}}$ whose cost $c \cdot x^{\texttt{int}} \le 10\gamma(c \cdot x)$ and $x^{\texttt{int}} \le 10d$.*

Proof. Let x be a feasible solution to $A[s]x \ge b, x \ge 0$. We construct an integral solution $x^{\texttt{int}}$ such that $A[s]x^{\texttt{int}} \ge b$ and $c^T x^{\texttt{int}} \le 10\gamma c^T x$. Let s_{max} and s_{min} be the largest and smallest s_j's.

Grouping: Let $\mathcal{C}^{(t)} := \{j : 2^{-(t+1)}s_{max} \ < s_j \le \ \ 2^{-t}s_{max}\}$ for $t = 0, 1, \ldots T$ where $T = \log(\frac{s_{max}}{s_{min}})$. Let $b_i^t := \sum_{j=1}^n A_{ij}s_jx_j$. Note that $\sum_{t=0}^T b_i^t \ge b_i$. Let $m_i^t := \min_{j\in\mathcal{C}^{(t)}:A_{ij}\ne 0} s_jA_{ij}$, that is, m_i^t is the smallest non-zero entry of the ith row of A in the columns of $\mathcal{C}^{(t)}$. Note that $m_i^t \ge 2^{-(t+1)}s_{max}$. Let m_i be the largest entry of row i. The assumption $A_{ij}s_j \le b_i$ implies $m_i \le b_i$.

Scaling: Let y^t be a vector with $y_j^t = 10x_j$ for $j \in \mathcal{C}^{(t)}$, 0 elsewhere. Note that $\sum_t c^T y^t = 10c^T x$ and $y_i^t \le 10d_i$ for any i. Let $\hat{s}^t$ be a vector with $\hat{s}_j^t = 2^{-(t+1)}s_{max}$ for $j \in \mathcal{C}^{(t)}$, 0 otherwise. Since for all $j \in \mathcal{C}^{(t)}$, $\hat{s}_j^t \ge s_j/2$, for all rows i we have

$$\sum_{j\in\mathcal{C}^{(t)}} A_{ij}\hat{s}_j^t y_j^t \ge 5\sum_{j\in\mathcal{C}^{(t)}} A_{ij}s_jx_j = 5b_i^t$$

Therefore since $m_i^t \ge 2^{-(t+1)}s_{max}$, we get

$$\sum_{j\in\mathcal{C}^{(t)}} A_{ij}y_j^t \ge \frac{5b_i^t}{2^{-(t+1)}s_{max}} \ge \frac{5b_i^t}{m_i^t} \ge \lfloor\frac{5b_i^t}{m_i^t}\rfloor$$

If we define an integral vector a^t to be $a_i^t := \lfloor\frac{5b_i^t}{m_i^t}\rfloor$, we see that $Ay^t \ge a^t$.

Using assumption 1, there exists an integral solution z^t such that $Az^t \ge a^t$, and $c^Tz^t \le \gamma(c^Ty^t)$, and $z_i^t \le 10d_i$.

Scaling back: Now fix a row i, and look at

$$\sum_{j \in \mathcal{C}^{(t)}} A_{ij} s_j z_j^t \geq \sum_{j \in \mathcal{C}^{(t)}} A_{ij} m_i^t z_j^t = m_i^t \sum_{j \in \mathcal{C}^{(t)}} A_{ij} z_j^t \geq m_i^t \lfloor \frac{5b_i^t}{m_i^t} \rfloor$$

where the first inequality follows since m_i^t is the minimum entry in the ith row in the columns of $\mathcal{C}^{(t)}$. This is where our analysis slightly differs from the previous analyses of grouping and scaling, where instead of multiplying the RHS by m_i^t, the RHS was multiplied by $2^{-t}s_{max}$. This subtle observation leads us to make a weaker assumption than the no-bottleneck assumption.

Getting the final integral solution:
Define $x^{\mathbf{int}} := \sum_{t=0}^{T} z^t$. Note that $c^T x^{\mathbf{int}} = \sum_t c^T z^t \leq \gamma \sum_t c^T y^t = 10\gamma(c^T x)$ and $x^{\mathbf{int}} \leq 10d$.

Fix a row i and look at the ith entry of $A[s]x^{\mathbf{int}}$.

$$\sum_{t=0}^{T} \sum_{j \in \mathcal{C}^{(t)}} A_{ij} s_j z_j^t \geq \sum_{t=0}^{T} \lfloor \frac{5b_i^t}{m_i^t} \rfloor m_i^t \tag{15}$$

Let $S_i := \{t : 5b_i^t < m_i^t\}$. Note that

$$\sum_{t \in S_i} b_i^t < \frac{1}{5} \sum_{t \in S_i} m_i^t \leq 3m_i/5$$

the second inequality following from Claim 3 below. This gives us

$$\sum_{t \notin S_i} b_i^t > \sum_{t=0}^{T} b_i^t - 3m_i/5 \geq b_i - 3m_i/5$$

For $t \notin S_i$, we have the floor in the inequality (15) at least 1. So we can use the relation $\lfloor x \rfloor \geq x/2$ for $x \geq 1$. Thus, using $m_i \leq b_i$, we have

$$Ax^{\mathbf{int}} \geq \sum_{t \notin S_i} \frac{5b_i^t}{2} \geq 5b_i/2 - 3m_i/2 \geq b_i$$

Claim 3. $\sum_{t=0}^{T} m_i^t \leq 3m_i$.

Proof. Note that the non-zero m_i^t decreases as t goes from 0 to T. Also, for any $t < t'$, we have $m_i^t > 2^{-(t+1)} s_{max}$ and $m_i^{t'} \leq 2^{-t'} s_{max}$. Thus, $m_i^{t'} \leq m_i^t \cdot 2^{-(t'-t-1)}$. Since the largest m_i^t can be at most m_i, $\sum_{t=0}^{T} m_i^t \leq m_i + m_i + m_i/2 + m_i/4 + ... \leq 3m_i$. □

□

3 Priority line cover

Recall the PLC problem where a segment j covers an edge e iff it contains it and $s_j \geq \pi_e$. We first show that the integrality gap of the canonical linear programming relaxation of PLC is at least $3/2$ and at most 2. Subsequently, we present an exact combinatorial algorithm for the problem.

$$\min\Big\{\sum_{j\in\mathcal{S}} c_j x_j : \quad x \in R_+^{\mathcal{S}} \qquad \text{(Primal)}$$
$$\sum_{j\in\mathcal{S}: j \text{covers } e} x_j \geq 1, \quad \forall e \in E\Big\}$$

$$\max\Big\{\sum_{e\in E} y_e : \quad y \in R_+^{E} \qquad \text{(Dual)}$$
$$\sum_{e\in E: j \text{covers } e} y_e \leq c_j, \quad \forall j \in \mathcal{S}\Big\}$$

Figure 1: The PLC canonical LP relaxation and its dual.

3.1 Canonical LP relaxation: Integrality gap

We start with the canonical LP relaxation for PLC and its dual in Figure 1.

The following example shows that the integrality gap of (Primal) is at least 3/2.

Example 2. Figure 2 shows a line of odd length k; odd numbered edges have demand 1, and even numbered edges have a demand of 2. Paths are shown as lines above the line graph, and are also numbered. Odd numbered paths have a supply of 2, and even numbered ones have a supply of 1. Dashed lines indicate edges spanned but not covered. All paths have cost 1. Note that a fractional solution is obtained by letting $x_p = 2/3$ for paths 2 and k, and $x_p = 1/3$ otherwise. The cost of this solution is $(k+3)/3$, while the best integral solutions takes all odd-numbered paths, and has cost $(k+1)/2$. As k tends to ∞, the ratio between the integral and fractional optimum tends to 3/2. As an aside, we found the above integrality gap instance by translating a known integrality-gap instance of the tree-augmentation problem in caterpillar graphs; see [8].

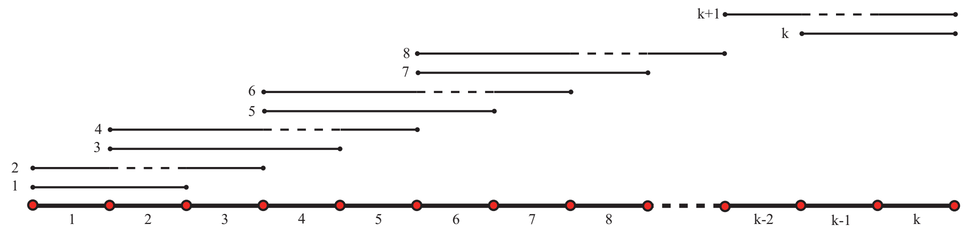


Figure 2: Integrality Gap for PLC

We now show that the integrality gap of the canonical LP for PLC is bounded by 2. We describe a simple primal-dual algorithm which constructs a feasible line cover solution and a feasible dual solution, and the cost of the former is at most twice the value of the dual solution.

The algorithm maintains a set of segments Q. Call an edge e *unsatisfied* if no segment in Q covers e. Let U be the set of unsatisfied edges. Initially Q is the empty set and $U = E$. We grow duals y_e on certain edges, as specified below. We let E_+ denote the edges with positive y_e; we call such edges, *positive* edges. Initially E_+ is empty. Call a segment j *tight* if $\sum_{e\in j: j \text{ covers } e} y_e = c_j$. We use the terminology an edge e is larger than f, if $\pi_e \geq \pi_f$.

Primal-Dual Algorithm

1. While U is not empty do
   - Breaking ties arbitrarily, pick the largest edge e in U.
   - Increase y_e till some segment becomes tight. Note that each such segment must contain e. Let $j_l(e)$ and $j_r(e)$ be the tight segments which have the smallest left-end-point and the largest right-end-point, respectively. Since e is chosen to be the largest uncovered edge, any unsatisfied edge contained in the two segments $j_l(e)$ or $j_r(e)$ is also covered. We say e is responsible for $j_l(e)$ and $j_r(e)$.

     Add $j_l(e), j_r(e)$ to Q. Add e to E_+. Remove all the unsatisfied edges contained in either $j_l(e)$ or $j_r(e)$ from U.
2. **Reverse Delete:** Scan the segments j in Q in the reverse order in which they were added, and delete j if its deletion doesn't lead to uncovered edges.

It is clear that the final set Q is feasible. It is also clear that y forms a feasible dual. The factor 2-approximation follows from the following lemma by a standard relaxed complementary slackness argument, and this finishes the proof of Theorem 3.

Lemma 5. *Any edge $e \in E_+$ is covered by at most two segments in Q.*

Proof. Suppose there is an edge $e \in E_+$ covered by three segments j_1, j_2 and j_3. Observe that one of the segments, say j_2, must be completely contained in $j_1 \cup j_3$. Since j_2 is not deleted from Q, there must be an edge $f \in j_2$ such that j_2 is the only segment in Q covering f. Since j_1 and j_3 don't cover f, but one of them, say j_1 contains it, this implies $\pi_f > \sup_{j_1} \geq \pi_e$. That is, f is larger than e.

If f is the edge responsible for j_2, then since j_2 contains e, e wouldn't be in E_+. Since f is larger than e, there must be a segment j in Q added before j_2 which covers f. In the reverse delete order, j_2 is processed before j. This contradicts that j_2 is the only segment in Q covering f. □

Lemma 6. $\sum_{j \in Q} c_j \leq 2 \sum_{e \in E} y_e$.

Proof. Since each $s \in Q$ satisfies $\sum_{e \in j: j \text{ covers } e} y_e = c_j$, we get

$$\sum_{j \in Q} c_j = \sum_{j \in Q} \sum_{e \in j: j \text{ covers } e} y_e = \sum_{e \in E} y_e \cdot |\{j \in Q : j \text{ covers } e\}| \leq 2 \sum_{e \in E} y_e$$

□

3.2 An Exact Algorithm for PLC

We first describe the sketch of the algorithm; the full proof starts from Section 3.2.1. A segment j covers only a subset of edges it contains. We call a contiguous interval of edges covered by j, a *valley* of j. The uncovered edges form *mountains.* Thus a segment can be thought of as forming a series of valleys and mountains.

Given a solution $S \subseteq \mathcal{S}$ to the PLC (or even a PTC) instance, we say that segment $j \in S$ is *needed* for edge e if j is the unique segment in S which covers e. The set of needed edges is denoted as $E_{S,j}$. We say a solution is *valley-minimal* if it satisfies the following two properties: (a) If a segment j is needed for edge e which lies in the valley v of j, then no higher supply segment of S intersects this valley v, and (b) every segment j is needed for its last and first edges. We show that an optimum solution can be assumed to be valley-minimal, and thus it suffices to find the minimum cost valley-minimal solution.

The crucial observation is the following. Let segment j be the one in the optimum solution which covers the first edge of the line. This decomposes the solution into two parts. Every other segment of the solution lies *completely* in one of two regions. Either it lies in the strict interior of the segment j, or it lies in the region from the right end-point of j to the right end-point of the line. This allows us to obtain the optimal substructure for PLCs. The second region's solution is a smaller PLC instance, and we show that the solution to the first region can be cast as a shortest path in a network whose arcs correspond to various smaller subproblems of PLC. The algorithm follows by dynamic programming.

3.2.1 Valley-Minimal Solutions

As mentioned above, it helps to think of supplies and demands as *heights*. In the case of PLC, the demands of the edges in E form a terrain, and each segment $j \in \mathcal{S}$ corresponds to a straight line at height s_j. Segment j then covers edge e if e lies in the segment's *shadow*, that is, the height of e is smaller than the height of the segment.

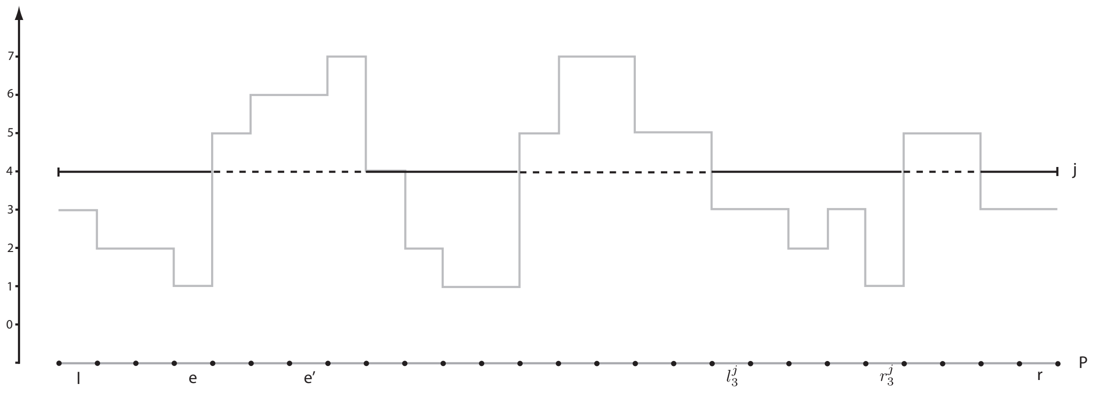


Figure 3: The figure shows a segment j, and the terrain induced by the edges of E that it contains. The terrain partitions j into valleys and mountains. Valleys are indicated by solid parts of j, and mountains are shown as dashed lines.

Figure 3 illustrates this with path P and its edges. The light gray terrain indicates the demands of the edges. The segment j shown in the picture covers the edges in $[l, r]$ that lie in its shadow; e.g., j covers edge e but not e'. The terrain partitions j naturally into *valleys* – contiguous sub-intervals of $[l, r]$ that are in the shadow of j, and *mountains* – those sub-intervals that are contained in $[l, r]$ and consist entirely of edges that are not covered by j. The parts of j that correspond to mountains are indicated by dashed lines, and valleys are depicted by solid lines. In the following, we let $[l^j_k, r^j_k]$ be the interval corresponding to the kth valley of j.

In the following, we will assume that the set of segments $\mathcal{S}$ in the given PLC/PTC instance is *segment-complete*; i.e., if $\mathcal{S}$ contains the segment j then it also contains all proper sub-segments. For example, if a PLC instance contains segment j corresponding to interval $[l^j, r^j]$, then it also contains segments corresponding to intervals $[l, r]$ for all $l^j \leq l \leq r \leq r^j$. This assumption is w.l.o.g. as we can always add a *dummy* sub-segment j' for any such interval $[l, r]$ with the same supply and cost as j. Any minimal solution clearly uses at most one of j and j', and if j' is used, then replacing it with j does not affect feasibility.

Let $S \subset \mathcal{S}$ be an inclusion-wise minimal solution for the given instance, and let $j \in S$ be any one of its segments. We say that j is *needed* for edge $e \in E$ if j covers e, and if there is no other segment in S that covers e; let $E_{S,j}$ be the set of edges that need j, and hence $E_{S,j} \neq \varnothing$ for all $j \in S$. Thus, if j is needed for e, then e is in one of j's valleys; we let val^j_e be that valley.

A solution $S \subseteq \mathcal{S}$ is *valley-minimal* if

[M1] for all $j \in S$ and for all $e \in E_{S,j}$, no segment of higher supply in S covers any of the edges in val^j_e, and

[M2] each segment is needed for its first and last edge.

For a solution $S \subseteq \mathcal{S}$, we say that (j, j', e) is a *violating triple* if $j, j' \in S$, j' has higher supply than j, j is needed for e, and j' covers some edge in val^j_e. We obtain the following observation.

Lemma 7. *Given a feasible instance of PLC/PTC, there exists an optimum feasible solution that is valley-minimal.*

Proof. First, it is not too hard to see that we can always obtain an optimal solution that satisfies [M2]. If S is an optimum solution with a segment j, and j is not needed for its first or last edge e, then we may clearly replace j by the sub-segment $j - e$. This does not increase the solutions cost, using the segment-completeness.

Assume, for the sake of contradiction that S violates [M1], and choose a solution S with the smallest number of violating triples. Let (j, j', e) be one such triple. Since j is needed for e, edge e is not contained in j', and hence j' is either fully contained in the interval $(e, n]$ or fully contained in the interval $[1, e)$. Using the segment-completeness assumption, we may replace j' by the sub-segment j'' obtained by removing the prefix consisting of edges in val^j_e; remove j'' if it is empty. The resulting set of segments has cost at most that of S, and the number of violating triples is smaller; a contradiction. □

In the next subsection, we show how we can compute the minimum cost valley-minimal solution for PLC instances in polynomial time using dynamic programming.

3.2.2 Computing valley-minimal solutions

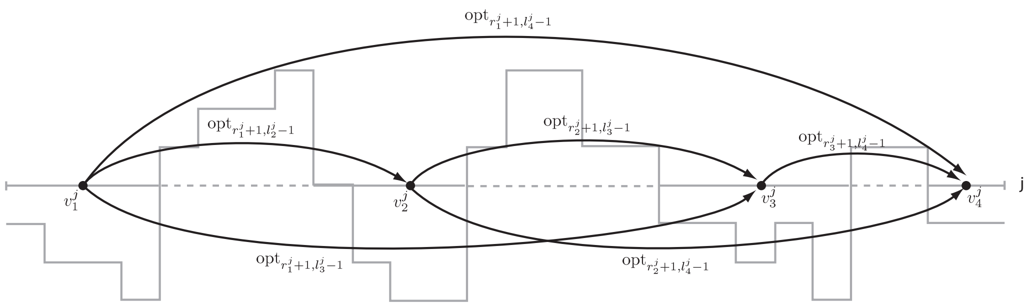
For $1 \leq l \leq r \leq n$, let $\mathtt{OPT}_{l,r}$ be the minimum cost valley-minimal feasible solution for the sub-instance induced by interval $[l, r]$ (i.e., this is the instance obtained by keeping only the segments that are entirely contained in $[l, r]$), and let $\mathtt{opt}_{l,r}$ be its cost. Note that the segment-completeness assumption implies that any such sub-instance is feasible. Clearly, $\mathtt{opt}_{n,n}$ is the minimum cost of any segment in $\mathcal{S}$ that covers edge n, and $\mathtt{OPT}_{1,n}$ is the optimum solution we want to obtain.

The following observation allows us to prove the optimal substructure in the problem for PLC instances. Let S be a valley-minimal solution for the sub-interval $[l, r]$, and let $j \in S$ be the unique (by valley-minimality) segment covering the first edge $(l, l+1)$. Let

$$E_{S,j} = \mathrm{val}^j_1 \cup \ldots \cup \mathrm{val}^j_k \tag{16}$$

be the set of edges in E that need j. Let $\mathrm{val}^j_i = [l^j_i, r^j_i]$ be the ith valley in the above list. In the following observation, we let $l^j_{k+1} = r$.

Observation 1. *We may assume, for all $1 \leq i \leq k$, if $j' \in S$ contains $e \in (r^j_i, l^j_{i+1})$, then j' is fully contained in (r^j_i, l^j_{i+1}).*



$\text{opt}_{r_1^j+1,l_4^j-1}$
$\text{opt}_{r_1^j+1,l_2^j-1}$
$\text{opt}_{r_2^j+1,l_3^j-1}$
$\text{opt}_{r_3^j+1,l_4^j-1}$
v_1^j
v_2^j
v_3^j
v_4^j
j
$\text{opt}_{r_1^j+1,l_3^j-1}$
$\text{opt}_{r_2^j+1,l_4^j-1}$

Figure 4: The part of digraph G_l corresponding to segment $j \in \mathcal{S}_l$.

Proof. If segment j' has higher supply than j, then the above follows from the first condition of valley-minimality. If segment j' has lower supply, then since j' must be needed for some edge e, j must not contain e implying j' must have its right end-point in (r_k^j, r). Replacing j' by the sub-segment (r_k^j, r) completes the observation. $\square$

Now we show how to compute $\texttt{OPT}_{l,r}$ given $\texttt{OPT}_{l'r'}$ for all $l \leq l' \leq r' \leq r$. The high level idea is the following. The algorithm guesses the first segment j in $\texttt{OPT}_{l,r}$. Given $j = (l, r')$ say, from the above observation the problem decomposes into two parts. One part is the interval $(r', r]$ which are covered by segments completely lying in sub-interval $(r', r]$), the other are the uncovered edges in (l, r') which are covered by segments completely lying in sub-interval (l, r'). The first part's solution is obtained since it is a smaller subproblem, the second part is computed by a shortest-path computation. We now elaborate and give the complete algorithm.

Let $\mathcal{S}_l$ be the segments in $\mathcal{S}$ with leftmost endpoint l. We construct a digraph G_ℓ as follows. Consider a segment $j \in \mathcal{S}_l$, and let

$$[l_1^j, r_1^j], \ldots, [l_k^j, r_k^j]$$

be the set of its valleys. We add a node v_q^j for each valley $1 \leq q \leq k$ of j to G_ℓ. We also add an arc for all $1 \leq q < q' \leq k$.

A shortest path corresponding to the solution $\texttt{OPT}_{l,r}$ will use arc $(v_q^j, v_{q'}^j)$ if

(i) j is the leftmost segment in $\texttt{OPT}_{l,r}$, and

(ii) val_q^j and $\text{val}_{q'}^j$ are two consecutive valleys of j for which j is needed. Note that j might not be needed for all of its valleys. However, by valley minimality, it is needed for its first and last valleys.

Observation 1 then states that $\texttt{OPT}_{l,r}$ uses segments that are entirely contained in $(r_q^j, l_{q'}^j)$ to cover $(r_q^j, l_{q'}^j)$. An optimum set of such segments is given by $\texttt{OPT}_{r_q^j+1, l_{q'}^j-1}$, and we therefore give arc $(v_q^j, v_{q'}^j)$ cost $\texttt{opt}_{r_q^j+1, l_{q'}^j-1}$. Figure 4 shows the part of G_ℓ for the segment s from Figure 3.

We add a source node $\bar{v}_j$ and arcs $(\bar{v}_j, v_1^j)$ of cost c_j for each of the segments $j \in \mathcal{S}_l$. A shortest path uses such an arc if j is the unique segment starting at l in the corresponding optimum solution.

We also add a sink node $\bar{t}_r$ and add an arc $(v_k^j, \bar{t}_r)$ for all $j \in \mathcal{S}_l$ of cost $\texttt{opt}_{r_k^j+1, r}$ indicating the optimum PLC for the sub-interval $[r_k^j + 1, r]$. Note that if $r_k^j = r$, then this arc is a loop of cost 0 and can be discarded.

It follows from the above construction that $\texttt{opt}_{l,r}$ is equal to the cost of a shortest $\bar{s}, \bar{t}_r$-path in G_l. Each of the shortest-path computations can clearly be done in polynomial time, and we therefore

$\texttt{opt}_{l,r}$ can be obtained via dynamic programming, in polynomial time. We get the following theorem which is a restatement of Theorem 5.

Theorem 10. *The cost* $\texttt{opt}_{1,n}$ *of an optimum solution for a given PLC instance can be computed in polynomial time.*

4 Priority tree cover

We first give a proof of Theorem 6, and show that rooted PTC is APX-hard, even if all segments have unit cost. Subsequently, we present a 2-approximation algorithm for the problem, by reducing it to an auxiliary instance of the tree augmentation problem. Then, we prove Theorem 4, and show that the integrality gap of the canonical LP formulation of unweighted PTC is bounded by 6. Finally, we prove the connection between PTC and the rectangle cover problem.

4.1 APX-hardness

We prove APX-hardness of PTC via a reduction from vertex cover in bounded degree graphs. The latter problem is known to be APX-hard. Given a bounded degree graph $G(V, E)$, with n vertices and $m = O(n)$ edges, let the edges be arbitrarily numbered $\{1, 2, \ldots, m\}$.

The tree in our instance has a broom structure: it has a *handle*, a path of m edges $(e_1, \ldots, e_m)$ given by vertices $\{x_0, x_1, \ldots, x_m\}$, and it has n *bristles*, one bristle corresponding to each vertex $v \in V$. The edge e_i in the handle for $1 \le i \le m$, corresponds to the edge numbered i in the graph G. The bristle corresponding to vertex v is a path $(f_1^v, f_2^v, \ldots, f_{deg(v)}^v)$ of length $deg(v)$ given by the vertices $\{x_m, y_1^v, y_2^v, \ldots, y_{deg(v)}^v\}$. The root of the tree is x_0, the end point of the handle. Thus the tree has $m + \sum_v deg(v) = 3m$ edges.

We now describe the priority demands of these tree edges. The demand of edge e_i is i. Consider the edges in G incident on v in the decreasing order of their numbers. Suppose they are $(i_1 > i_2 > \cdots > i_{deg(v)})$. The demands of the edge f_j^v is i_j. Thus, for a particular bristle corresponding to a vertex v, the demands decrease as we go from f_1^v to $f_{deg(v)}^v$, and these demands correspond to the numbers of edges incident on v.

Now we describe the segments. All segments have unit cost. We have two kinds of segments: edge segments and vertex segments. For every edge $i = (v, w)$ in E, there are two edge segments s_v^i and s_w^i. Segments s_v^i contains all edges e_i to e_m and edges f_1^v to f_j^v, where edge i is the jth edge in the descending order of neighbors of v in G. The supply of segment s_v^i is i, and thus by construction, we see that s_v^i only spans edge e_i and f_j^v. That completes the description of edge segments. For every vertex v, there is a vertex segment t_v which covers all the edges in the bristle corresponding to vertex v. That completes the description of the PTC instance. Look at figure 5 for an illustration of the reduction.

The following lemma along with the APX-hardness of the vertex cover problem in bounded degree graphs, and the fact that in the latter any vertex cover is of size $\Omega(n)$, leads to the APX-hardness of the PTC problem.

Lemma 8. *The optimum PTC of the above instance is* $m + k$*, where* k *is the size of the optimum vertex cover of* G*.*

Proof. Firstly note that we may assume that in any optimal PTC, for any edge $i = (v, w)$, we will have exactly one of s_v^i or s_w^i in the solution. We need to have one since these are the only two segments which cover edge e_i in the tree. Instead of picking both, we can remove one, say s_w^i, from

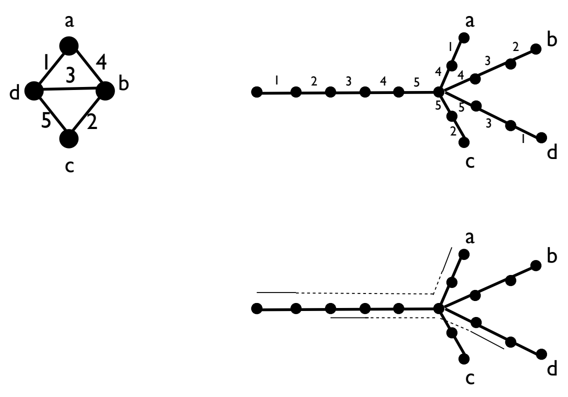


Figure 5: The graph on the left is the instance of the vertex cover problem, the tree on the right is the corresponding PTC instance. The numbers on the edges are the priority demands corresponding to the edge numbers in the graph. In the second figure to the right, we illustrate two segments: s_a^1 and s_d^3, having supplies 1 and 3 respectively. Dashed line means that these segments do not have enough supply to cover the edges.

the solution and pick the corresponding vertex segment t_w instead, at no increase of cost. Therefore, there are exactly m edge segments picked in any optimal PTC solution.

Now note that these m edge segments uniquely correspond to an orientation of the edges in G; if for edge $i = (v, w)$, s_v^i is chosen in the solution, the edge (v, w) is oriented from w to v. In this orientation, if there is a *sink* (a vertex with all edges incident to it) v, then note that all the edges in the bristle corresponding to v have also been covered. Thus, the number of vertex segments required to cover the remaining edges of the tree, is precisely the number of *non-sinks* in this orientation. In particular, the optimal PTC corresponds to the orientation which minimizes the number of non-sinks.

The proof is complete by noting that non-sinks form a vertex cover; this is because each edge is oriented away from some non-sink, and is thus incident to it. Furthermore, given a vertex cover, there exists an orientation with precisely these vertices as non-sinks. Orient the edges towards the complement of the vertex cover (the independent set) - the complement is precisely the set of sinks, and thus the vertex cover is precisely the set of non-sinks. □

Proof of Theorem 6 Suppose the degrees of G are all B, a constant. Note that the vertex cover of this graph is at least $m/B = n/2$. The APX-hardness implies that it is NP-hard to distinguish between the case when the vertex cover is $c_1 n$ or $c_2 n$ where $c_2 > c_1 \geq 1/2$ are certain constants.

The above lemma therefore implies it is NP-hard to distinguish between the cases when the optimum of a PTC is $m + c_1 n = (c_1 + B/2)n$ and when the optimum is $m + c_2 n = (c_2 + B/2)n$. Since B, c_1, c_2 are constants, we get the APX-hardness.

(For the interested reader: the APX-hardness of vertex cover of bounded degree graphs by Berman and Karpinski [3] gives $B = 4$, $c_1 = 78/152$ and $c_2 = 79/152$, showing it is NP-hard to approximate to a factor better than 1.002.) □

4.2 An approximation algorithm for PTC

The crucial idea is the following. Given an optimum solution $S^* \subseteq \mathcal{S}$, we can partition the edge-set E of T into disjoint sets $E_1, \ldots, E_p$, and partition two copies of S^* into $S_1, \ldots, S_p$, such that E_i is a path in T for each i, and S_i is a priority line cover for the path E_i.

In particular, we prove the following lemma. Let $\hat{E}_{S^*,j}$ be the set of edges e such that j is the segment with the highest supply, among all segments in S^* which cover e. Note that the union of all $\hat{E}_{S^*,j}$, over all $j \in S^*$, partitions E. Also note that for each edge e, there is a unique segment j such that $e \in \hat{E}_{S^*,j}$. We call the segment j *responsible* for e.

Lemma 9. *Given a valley-minimal solution $S^* \subseteq \mathcal{S}$ to a PTC instance with tree $T = (V, E)$, there is a partition*

$$E_1 \cup \ldots \cup E_p = E,$$

where each E_i is the edge set of a path in T such that $\hat{E}_{S^*,j} \cap E_i \neq \varnothing$ for at most two $i \in \{1, \ldots, p\}$, for all $j \in S^*$.

Using this, we describe the 2-approximation algorithm which proves Theorem 7.

Proof of Theorem 7 For any two vertices t (top) and b (bottom) of the tree T, such that t is an ancestor of b, let P_{tb} be the unique path from b to t. Note that P_{tb}, together with the restrictions of the segments in $\mathcal{S}$to P_{tb}, defines an instance of PLC. Therefore, for each pair t and b, we can compute the optimal solution to the corresponding PLC instance; let the cost of this solution be c'_{tb}. Create an instance of the 0,1-tree cover problem with T and segments $\mathcal{S}' := \{(t, b) : t \text{ is an ancestor of } b\}$ with costs c'_{tb}. Solve the 0,1-tree cover instance exactly (recall we are in the rooted version) and for the segments (t, b) in $\mathcal{S}'$ returned, return the solution of the corresponding PLC instance of cost c'_{tb}.

We now use Lemma 9 to obtain a solution to the 0,1-tree cover problem $(T, \mathcal{S}')$ of cost at most 2 times the cost of S^*. The segments in $\mathcal{S}'$ picked are precisely the segments corresponding to paths E_i, $i = 1, \ldots, p$. We now use the lemma to find $S_1, \ldots, S_p$ such that each S_i is a PLC for E_i, $S_i \subseteq S^*$ and each segment in S^* is in at most two S_i. Since the cost of each of these S_i will be more than the cost of the optimum PLC which our algorithm finds, this will complete the proof.

Define $S_i := \{j \in S^* : e \in E_i \cap \hat{E}_{S^*,j}\}$ to be the set of segments responsible for the edges in E_i. By definition, S_i is a PLC for E_i. By the lemma, a segment $j \in S^*$ lies in at most two S_i. □

Proof. (Lemma 9) We give an algorithm to compute the decomposition. Let e be any of the edges incident to the root of T, and let $j_1 \in S^*$ be the highest-supply segment covering e. We then let E_1 be the edges of the path in T corresponding to j_1. Removing E_1 from T yields sub-trees $T_1, \ldots, T_q$. For each tree T_i we repeat the above steps, and let

$$E_1, \ldots, E_p$$

be the final partition; let $j_i \in S^*$ be the segment corresponding to edge-set E_i.

Consider a segment $j \in S$, and let $1 \leq i \leq p$ be smallest such that $E_{S^*,j} \cap E_i \neq \varnothing$, and assume that $E_{S^*,j} \cap E_q \neq \varnothing$ for some $i < q \leq p$; choose q smallest with this property. We claim that $j_q = j$, and hence there cannot be a $q < q' \leq p$ with $E_{S^*,j} \cap E_{q'} \neq \varnothing$.

Let $e \in E_{S^*,j} \cap E_i$, and let $f \in E_{S^*,j} \cap E_q$ be two edges in different parts of the partition that need segment j. As both e and f are edges on j, and since $i < q$, it follows that f is a descendant of e in tree T. Let g be the topmost edge of E_q; clearly, g is on the e, f-path in T. By the decomposition algorithm, segment j_q is the highest-supply segment covering edge g. As j contains g, this means that the supply of j_q is at least that of j. Finally, since f is on j_q, j_q covers f as well. But this means that $j_q = j$ as j is needed by f. □

4.3 Canonical LP relaxation of PTC: Integrality Gap

In this section, we prove Theorem 4, by showing that the canonical LP relaxation of unweighted PTC is at most 6. Recall the PTC LP.

$$\min \quad \left\{ \sum_{s \in \mathcal{S}} c_s x_s : \quad \forall e \in E : \sum_{s:s \ covers \ e} x_s \geq 1; \quad x_s \geq 0, \forall s \in \mathcal{S} \right\} \tag{17}$$

Proof of Theorem 4. The idea of the proof is the following: as in the factor 2-approximation for PTC, we decompose the edge set of the tree into disjoint sets $E_1, \cdots, E_p$, such that each E_i induces a path. We will abuse notation and refer to the E_i's as paths. Furthermore, we take any feasible solution x of (17) and obtain p fractional solutions $x^{(1)}, \ldots, x^{(p)}$ such that $x^{(i)}$ is a feasible fractional solution to (Primal) for the PLC instance on the path E_i. We will guarantee that

$$\sum_{i=1}^{p} \sum_{j \in \mathcal{S}} x_j^{(i)} \leq 3 \sum_{j \in \mathcal{S}} x_j.$$

The theorem then follows from Theorem 3.

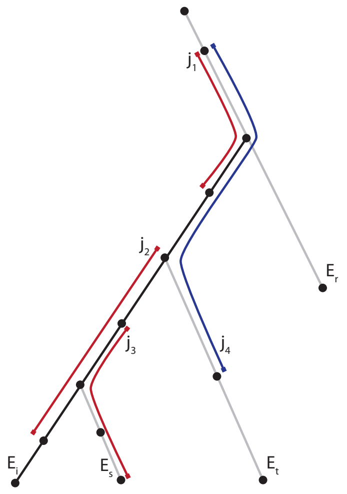


Figure 6: The figure shows a fragment E_i, its parent E_r, and two children E_s and E_t. The segments j_1, j_2, and j_3 are local, and segment j_4 is global. In particular, j_4 is an i, t-global segment.

Unlike in the argument used in the previous section where the decomposition into paths depended on S^*, the decomposition into disjoint paths that we use here is universal. Each path E_i will end at a unique leaf; that is p is the number of leaves of T. Let E_1 be *any* path from the root to a leaf. Delete E_1 from the tree to get a series of sub-trees. Recursively, obtain E_2 to E_p. We call a path E_i a *child* of E_q, if the starting point of E_i lies on E_q.

Let x be any feasible fractional solution of (17) and let S^* be the support of x, that is, $S^* = \{j : x_j > 0\}$. Fix a path E_i and say that a segment $j \in S^*$ *intersects* E_i if j covers an edge in E_i A segment j which intersects E_i is called *local* for E_i if either the first or the last edge covered by j lies in E_i. A segment j which intersects E_i is called *global* for E_i, otherwise. Figure 6 illustrates this.

Let j be a global segment for E_i, and let e be the first edge contained in j *after* E_i. If $e \in E_q$, we call s an *iq-global* segment. Observe that E_q is a child of E_i. Thus an iq-global segment *enters*

E_i and *exits* via E_q. Note that iq-global segments, over all q, partition all global segments for E_i. Also note that an iq-global segment could also be a $i'q'$-global segment for some other i', q'.

Now we are ready to define the fractional solution $x^{(i)}$ which will be feasible for (Primal) for the PLC instance on E_i. Firstly for all segments j which are local for E_i, let $x_j^{(i)} = x_j$. Next, we take care of segments which are global for E_i. Order all the iq-global segments in non-decreasing order of supply: $\{j_1, \ldots, j_r\}$. Let l be such that

$$x_{j_1} + \cdots + x_{j_l} \leq 1 \text{ and } x_{j_1} + \cdots + x_{j_l} + x_{j_{l+1}} > 1$$

If no such l exists, then $l = r$. Define $x_{j_k}^{(i)} = x_{j_k}$ for $1 \leq k \leq l$. If $l < r$, then let $x_{j_{l+1}}^{(i)} = 1 - \sum_{k=1}^{l} x_{j_k}^{(i)}$.

Claim 4. *$x^{(i)}$ is feasible for* (Primal) *for the PLC instance on E_i.*

Proof. Pick any edge $e \in E_i$. Look at all segments $j \in S^*$ which cover e. These segments are either local for e or global for e. If j is local, there is a corresponding segment in the support $x^{(i)}$ of the same value. Furthermore for any q,

$$\sum_{j:j \text{ is } iq\text{-global}, j \text{ covers } e} x_j^{(i)} \geq \min\{1, \sum_{j:j \text{ is } iq\text{-global}, j \text{ covers } e} x_j\}$$

In any case, e is covered by $x^{(i)}$ at least to the extent it is covered by x, which implies $x^{(i)}$ is feasible. □

Lemma 10. $\sum_{i=1}^{p} \sum_{j \in \mathcal{S}} x_j^{(i)} \leq 3 \sum_{j \in \mathcal{S}} x_j$

Proof. Each segment $j \in S^*$ is local for at most two paths E_i and E_q. Thus the contribution to the LHS by local segments for some path E_i is exactly $2 \sum_{j \in \mathcal{S}} x_j$.

Furthermore, for every parent-child pair E_i and E_q which induces an iq-global segment for E_i, we increase the LHS by at most 1. The number of such pairs is at most the number of leaves in T. The proof is complete by noting that $\sum_{j \in \mathcal{S}} x_j$ is at least the number of leaves in T. □

To complete the proof of the theorem, note that from Theorem 3 we know there exists for each E_i, a set of segments S_i such that $|S_i| \leq 2 \sum_{j \in \mathcal{S}} x_j^{(i)}$. The union of all such S_i forms a valid PTC of cardinality at most $6 \sum_{j \in \mathcal{S}} x_j$. □

4.4 Priority Tree Cover and Geometric Covering Problems

In this section, we show that the PTC problem is a special case of covering a set of points in 3-dimension by axis-parallel rectangles (cuboids). In particular we prove Theorem 8. We go in two steps. We first define a problem, which we call 2-Priority Line Cover and show that the PTC problem is a special case of 2-PLC. Subsequently, we show 2-PLC is a special case of 3-dimensional rectangle cover. We start with a definition of 2-PLC.

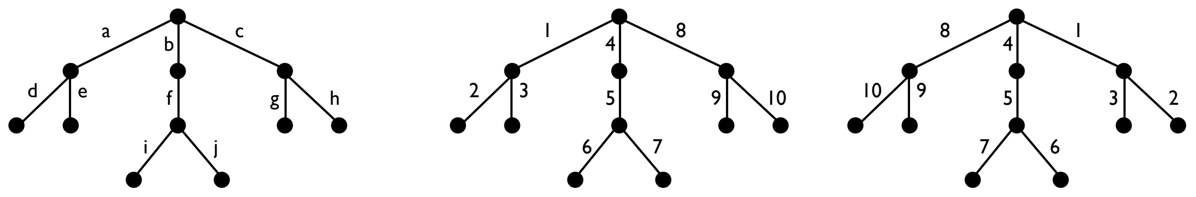
2-Priority Line Cover (2-PLC) The input is a line $T = (V, E)$, and a collection of segments $\mathcal{S} \subseteq V \times V$ with costs c_j for each $j \in \mathcal{S}$. Furthermore, each segment j has a priority supply vector in *two* dimensions, denoted as (s_j^1, s_j^2), and each edge e has a priority demand vector in two dimensions, denoted as (π_e^1, π_e^2). A segment j covers e iff j contains e and $s_j^i \geq \pi_e^i$ for both $i = 1, 2$. The goal is to find the minimum cost collection of segments which cover every edge.

It is easy to see that PLC is a special case of 2-PLC. Somewhat surprisingly, PTC is a special case of 2-PLC as well.

Lemma 11. *Any instance of PTC can be encoded as an instance of 2-PLC with the same solution set.*

Proof. Given a rooted tree $T = (V, E)$, we perform two different depth first traversals to get two different orderings on the edges E. One such ordering will define the line of the 2-PLC instance, the other will define the first coordinates of the priority demand vectors of the edges.

In a depth first traversal of a tree, at every step we move from a vertex to one of its children, if any. Our two different traversals will be defined by two different choices of moving to a children vertex. For every vertex v of the tree, consider a total order σ_v on its children. One such order which is convenient to keep in mind is the following; given a drawing of the tree, the total order of the children is from left to right. Let σ_v^R be the *opposite* total order. The two depth first traversals are obtained by running with σ_v's and σ_v^R's, respectively. Figure 7 illustrates the two orders with the ordering σ_v at every vertex v being from left-to-right, and σ_v^R being from right-to-left.



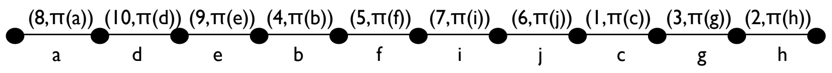


Figure 7: The left most tree is the original tree, the second and third are the two depth first traversals. The line below shows the line in the 2-PLC instance.

Let the two traversals return orderings μ and μ^R on the edges of the tree. The crucial observation is the following: for any vertex v, let $(v_1, \ldots, v_k)$ be the children in the σ_v order; then $\mu(v, v_1) < \mu(v, v_2) < \cdots < \mu(v, v_k)$, and thus, $\mu^R(v, v_1) > \cdots > \mu^R(v, v_k)$.

Now we are ready to describe the 2-PLC instance. The line is defined by the edges of the tree ordered w.r.t. μ. That is, the order of the edges is $(e_1, \ldots, e_m)$ such that $\mu(e_1) < \mu(e_2) < \cdots < \mu(e_m)$. The priority demand vector of an edge e of the tree is $(\mu^R(e), \pi_e)$. Consider a segment $j = (u, v)$ such that u is a child of v in the PTC instance. Let (u, u') and (v, v') be the unique edges incident to u and v respectively contained in j (in particular, u' is the parent of u and v' is the unique child of v which is an ancestor of u). By the depth-first property, we get $\mu(v, v') < \mu(u, u')$. The corresponding segment in the 2-PLC instance, also denoted as j, contains all the edges from $\mu(v, v')$ to $\mu(u, u')$. The priority supply vector of j is $(\mu^R(u, u'), s_j)$.

Claim 5. *For any segment j, the set of edges covered by j in the 2-PLC instance is precisely the set of edges covered in the PTC instance.*

Proof. Let e be an edge covered by j in the PTC instance. Since e is contained in the path from u to v in the tree, by property of depth first traversals we get, $\mu(v, v') < \mu(e) < \mu(u, u')$ and $\mu^R(e) < \mu^R(u, u')$. The first pair of inequalities implies e lies in the segment j in the 2-PLC instance, the second implies that $\pi_e^1 < s_j^1$. Since e is covered by j in the PTC, we also get $\pi_e^2 = \pi_e \leq s_j = s_j^2$. Thus, e is covered by j in the 2-PLC instance.

Let e be an edge covered by j in the 2-PLC instance. Since e lies in j, we conclude $\mu(v, v') < \mu(e) < \mu(u, u')$. This implies either (a) e lies on the path from u to v in the tree, or, (b) there is a node w on the u', v'-path in the tree, and a child z of w that is not on this path such that e is contained in the subtree defined by edge (w, z).

Note, that in case (b) the depth-first traversal for order σ visits edge (z, w) *before* edge (u, u'). This implies that the second dfs traversal for order σ^R visits (z, w) *after* (u, u'). Since (z, w) is visited before e in both traversals, we must therefore have $\mu^R(e) > \mu^R(u, u')$, and this implies $s_j^1 < \pi^1(e)$ which is impossible since j covers e. Thus, case (b) is not possible, and e lies on the path fro u to v on the tree. Furthermore, we have $s_j = s_j^2 \geq \pi_e^2 = \pi_e$, and so j covers e in the PTC instance as well. $\square$

$\square$

Now we show that 2-PLC is a special case of 3-dimensional rectangle cover. This is not to hard to see. We assume the edges of the line are numbered $(1, 2, \ldots, m)$. For edge e numbered e_i, we associate a point in 3 dimensions with coordinates (e_i, π_e^1, π_e^2). For each segment $j = (a, b)$, we have a rectangle associated. In fact, these rectangles have are unbounded in the negative y and z coordinates. The other 4 bounding half-spaces are $x \geq a$, $x \leq b$, $y \leq s^1(j)$ and $z \leq s^2(j)$. It is not too hard to see a rectangle corresponding to a segment j contains a point corresponding to an edge e iff j covers e in the 2-PLC instance. This completes the proof of Theorem 8.

5 Concluding Remarks

In this paper we studied column restricted covering integer programs. In particular, we studied the relationship between CCIPs and the underlying 0,1-CIPs. We conjecture that the approximability of a CCIP should be asymptotically within a constant factor of the integrality gap of the original 0,1-CIP. We couldn't show this; however, if the integrality gap of a PCIP is shown to be within a constant of the integrality gap of the 0,1-CIP, then we will be done. At this point, we don't even know how to prove that PCIPs of special 0,1-CIPS, those whose constraint matrices are totally unimodular, have constant integrality gap. Resolving the case of PTC is an important step in this direction, and hopefully in resolving our conjecture regarding CCIPs.

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