

Analytical time-optimal trajectories for an omni-directional vehicle

Weifu Wang and Devin J. Balkcom

Abstract—We present the first analytical solution method for finding a time-optimal trajectory between any given pair of configurations for a three-wheeled omni-directional vehicle in an obstacle-free plane. The mathematical model of the vehicle bounds the velocities of the wheels independently. The time-optimal trajectories can be divided into two categories: singular and generic. An analytical solution method has previously been presented for singular trajectories; this paper completes the work and presents the solution for generic trajectories. The speed and precision of the algorithm allow dense sampling of the configuration space, to determine how the time and structure of time-optimal trajectories change across configurations. Simulation results show that time-optimal trajectories tend to be ten to twenty percent faster than a simple but practical driving strategy: turn until the fastest translation direction faces the goal, drive to the goal, and turn to the current angle.

I. INTRODUCTION

This paper presents the first analytical method to find a time-optimal trajectory between any pair of configurations of a particular symmetric three-wheeled model of an omni-directional vehicle, without obstacles in the workspace. The vehicle, which is shown in figure ??, can move in any direction instantaneously. Its wheels are *omniwheels* – the wheels are powered in the directions that they are driven (v_1 , v_2 and v_3 in figure ??), but can slide freely in the perpendicular directions. We assume that the speeds of the wheels can be controlled, and that these wheel speeds are each bounded to fall in the range $[-1, 1]$ for some choice of units of distance.

The configuration of the robot is given by $q = (x, y, \theta)$, the location and orientation of a frame attached to the center of the robot. Although the wheel speeds are directly controlled by the motors, there is a one-to-one-mapping between wheel speeds and the generalized velocity of the robot in its own frame; we choose the generalized velocity as the control. The motion of the robot is given by

$$q(t) = q(0) + \int_0^t \mathcal{R}(\theta(\tau))u(\tau)d\tau, \quad (1)$$

where $u = ({}^R\dot{x}, {}^R\dot{y}, {}^R\dot{\theta})$ is the generalized velocity in the robot frame, and \mathcal{R} is the matrix that transforms the velocity into the world frame (formed by replacing the upper left block of a 3x3 identity matrix with a 2x2 rotation matrix).

The current paper builds heavily on previous work. The recent Ph.D. thesis of Andrei Furtuna [?] attacked a more-general problem: solving the time-optimal trajectories for a rigid body in the plane with polyhedral bounds on generalized velocity controls; the omni-directional robot problem is a special case. The thesis uses Pontryagin’s Maximum Principle (PMP) to derive strong necessary conditions on optimal trajectories, and proves further geometric and algebraic results.

There is a geometric interpretation of trajectories satisfying the Maximum Principle, which we will call extremal trajectories. For each extremal trajectory, there is a directed line in the plane (called the control line), such that the velocity of a point on the line, pushed by the robot, is maximized at almost every time. There are two types of trajectories – *generic*, for which the motion of the robot is completely characterized by the control line, and *singular*, for which it is not.

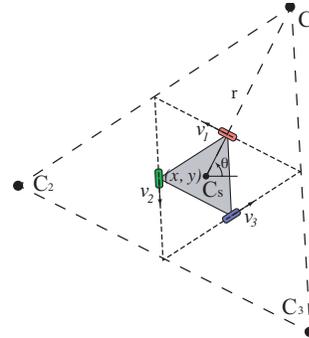


Fig. 1: The shape and the notation of the model



Fig. 2: A sample time-optimal trajectory

In [?], a complete analytical method is presented for solving for the time-optimal singular trajectory between any pair of configurations. However, until now, the problem of determining the precise generic trajectory that connects any two particular configurations has evaded solution. This solution is the primary contribution of the current paper.

We have studied this model in previous work [?], and in that work, derived a complete geometric description of the types of trajectories that might be optimal. We will use several results from [?]:

- 1) Although time and control space are continuous, most controls do not appear in optimal trajectories. In fact, trajectories with piecewise constant controls chosen from a set of 14 discrete controls are sufficient to achieve optimality for the bounded wheel speed model. The control set includes clockwise and counterclockwise *spins* in place about robot center C_s , forward or backward translation in three directions, and clockwise or counter-clockwise rotations (*turns*) about three points (C_1, C_2, C_3 in figure ??) placed in an equilateral triangle about the robot.
- 2) Generic trajectories are periodic, in the sense that for each trajectory there exists a duration T_p such that any two configurations along a trajectory separated by T_p represent effectively a pure translation parallel to a fixed line in the plane, the control line.

In this work, the basic approach is to compare *all* the extremal trajectories connecting given start and goal configurations to find a time-optimal one.

A. Related Work

Time-optimal trajectories for several other kinematic models have been found. A car that can only drive forwards with bounded forward velocity and steering angle was studied by Dubins [?]. Reeds and Shepp [?] found the shortest

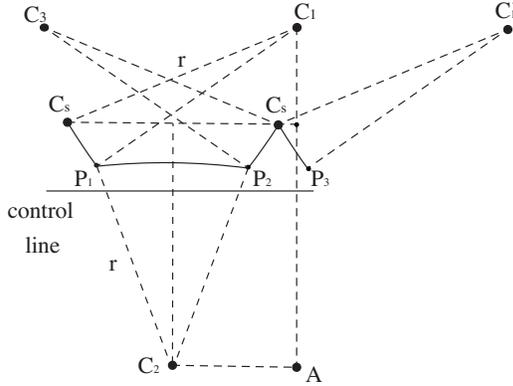


Fig. 3: Geometry of a shuffle trajectory

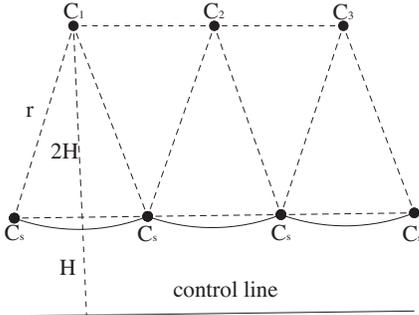


Fig. 4: Detailed roll trajectory.

paths for a steered car that can drive both forwards and backwards. Sussmann and Tang [?] further refined the results by developing a general methodology for solving this type of problem. Souères, Boissonnat and Laumond [?], [?] mapped the pairs of configurations to optimal trajectories for the Reeds-Shepp car. Balkcom and Maon [?] discovered the time-optimal trajectories for a differential-drive robot with bounds on the wheel speeds; Chitsaz *et al.* [?] determined the trajectories for a differential-drive that minimizes the sum of the rotation of the two wheels.

Dynamic models of omni-directional vehicle with or without slip have been developed by Jung and Kim [?], and Williams *et al.* [?]. Optimal trajectories for omni-directional vehicle using a dynamic model have been studied by Choi *et al.* [?] and Kalmár-Nagy *et al.* [?]. However, because of the complexity of the model, the resulting algorithms have been numerical rather than analytical, and assume certain restrictions on the structure of trajectories considered. The kinematic model we study is less sophisticated, but does admit complete analytical solution. Optimal motion of dynamic models for an underwater vehicle were studied by Chyba and Haberkorn [?]. A new distance function using nonholonomic constraints were used to find the minimum-time distance between a Dubins' car and polygonal obstacles [?], [?].

II. NECESSARY CONDITIONS FOR OPTIMALITY

We have shown in [?] that a necessary condition for time-optimal motion of a rigid-body with polyhedral bounds on generalized velocity controls (such as our omni-directional robot model), derived by application of Pontryagin's Principle [?] (PMP), is that for any time-optimal trajectory of the robot, there must exist constants k_1 , k_2 , and k_3 , not all zero, such that at any time along the trajectory, the generalized velocity of the robot $(\dot{x}, \dot{y}, \dot{\theta})$ must maximize the quantity

H , called the *Hamiltonian*, given by:

$$H = k_1\dot{x} + k_2\dot{y} + \dot{\theta}(k_1y - k_2x + k_3). \quad (2)$$

Furthermore, H must be constant and positive over the trajectory.

Without loss of generality, assume $k_1^2 + k_2^2 = 1$. (For this vehicle, we have shown optimality may be achieved without use of the $k_1 = k_2 = 0$ case, and non-zero scaling of the constants does not affect the maximization condition.) Let there be a line in the plane with direction given by the vector (k_1, k_2) ; let the signed distance of the line from the origin be k_3 . Call this line the control line.

For trajectories containing no translations (generic trajectories for the omni-directional robot model do not), the Hamiltonian equation can also be given by:

$$H = L_{y_c}\dot{\theta}, \quad (3)$$

where L_{y_c} is the distance of the current rotation center from the control line. The choice of constants determines the placement of the control line in the plane. Geometrically, we can say that for any optimal trajectory, we must be able to draw a particular line in the plane, such that at every time during the trajectory, the current rotation center, scaled by the angular velocity, is, among all possible rotation centers used to control the robot, as far as possible from the line.

Initial conditions of the robot relative the line determine the evolution of a generic extremal trajectory. There are two types, called *roll* and *shuffle* trajectories. Shuffle trajectories occur when the robot starts very close to the control line, with particular orientations. In this case, the trajectory period contains 4 rotations, and one of them has opposite direction compared to the others. Roll trajectories tend to occur when the robot starts far from the control line; in this case, all rotations are in the same direction.

In this paper, we choose the units such that the distance between each wheel and the center of the robot is 1, and the angular velocity of spin has magnitude 1. Then, each rotation center has distance 4 to the center of the robot. We describe each generic extremal with some notation indicating rotation center and sign of angular velocity. For example, one period of a *shuffle* trajectory might be of the form $C_s^+ C_1^+ C_2^- C_3^+ C_s^+$ in figure ??.

A. Time-optimal trajectories never exceed one period

A *switch* is a time during a trajectory at which the control changes. A *segment* of trajectory is the subsection of the trajectory between two switches.

Theorem 1: No time-optimal roll or shuffle trajectory for the omni-directional vehicle can be longer than one period. (One period of shuffle may contain at most four switches, and one period of roll may contain at most six switches.)

Proof: The result for shuffle trajectories has been previously proven in [?]. The proof for rolls can be divided into three cases. In each case, the method is, given a trajectory longer than one period, to construct a new trajectory that takes the same time, but does not satisfy the Pontryagin Principle. Since the new trajectory does not satisfy necessary conditions and cannot be optimal, the original equal-time trajectory also cannot be optimal.

Case 1: A period of a roll that starts and ends with spin, and the turn is longer than 60° . Construct a new trajectory by reflecting the 60° segments of each turn across the line passing P_1 and P_2 . This is trajectory $C_s C_1 C_1' C_1 C_s \dots$ in figure ?? . Denote $C_1 A = d_1$, and $C_1' A = d_2$. In this case, $H = \frac{1}{3}d_1 \neq \frac{1}{3}d_2$. So the new trajectory violates the necessary condition for optimality, because in this trajectory the Hamiltonian is not constant.

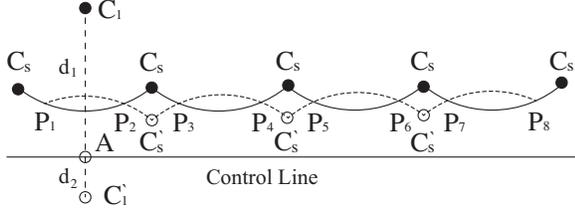


Fig. 5: Figure for proof that roll trajectory cannot be longer than one period.

Case 2: A period of a roll that starts and ends with spin or turns, in which the maximum turn is shorter than 60° . Reverse all the spins and replace turn controls with symmetric controls following the original path. Denote the distance from C_s to control line as d . Then the original $H = d = \frac{1}{3}d_1$, while the Hamiltonian for the reversed spin is $-d$. This new trajectory violates the necessary condition for optimality because the Hamiltonian is not constant during the trajectory.

Case 3: A period of roll that starts and ends with a turn longer than 60° . The initial point is either on the original curve between P_1P_2 or it is on P_2C_s . If it is on P_1P_2 , follow the rest of period by $C_1C_1'C_2C_2'C_3C_3' \dots$ till the end of the period. The arc length of P_3P_4 and P_4P_5 are 60° . Because the signed distance from C_s to the control line is $-d$, the Hamiltonian is not constant during the trajectory, violating the necessary conditions for optimality. If the initial point is on P_2C_s , reflect the 60° part $P_iP_{i+1}, i = 3, 5, 7$ across the line passing S_i s. This new trajectory also violates the necessary condition of optimality for the same reason as in Case 1.

III. ALGORITHM

Algorithm 1: Main algorithm flow

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Input:  $t_{\min} = t_{\text{singular}}$ 
for control  $i$  at initial do
  for control  $j$  at goal do
    if  $i$  is spin &&  $j$  is spin then
      Compute  $H$  (eq. ??), time  $t_r$  (eq. ??, ??)
      Compute  $H$  (eq. ??), time  $t_s$  (eq. ??, ??)
    if one of  $i$  and  $j$  is spin and the other is turn then
      Compute  $H$  (eq. ??), time  $t_r$  (eq. ??, ??, ??)
      Compute  $H$  (eq. ??), time  $t_s$ 
      (eq. ??, ?? or ??, ??) (2 cases)
    if  $i$  is turn &&  $j$  is turn then
      Compute  $H$  (eq. ??), time  $t_r$  (eq. ??, ??)
      Compute  $H$  (eq. ??, ??), time  $t_s$ 
      (eq. ??, ?? or ??, ??, ??) (2 cases)
     $t_{\min} = \min(t_{\min}, t_r, t_s)$ 

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The Hamiltonian is a constant over a time-optimal trajectory. We will see that for roll and shuffle trajectories, knowing the first and last control, as well as the start and goal configurations of the robot, this value of the Hamiltonian can be computed. Once the Hamiltonian has been computed, the complete evolution and time cost of the trajectory can be computed exactly.

Our algorithm loops over possible control choices for the first and last control of the trajectory, and for each choice computes a candidate Hamiltonian value and time cost. Equations for the computation of the Hamiltonian depend on whether the trajectory is a roll or shuffle, and whether the first and last controls are spins or turns; we will now consider each case.

A. Calculating time for each trajectory

Each trajectory that takes the robot from initial configuration to the goal configuration contains n segments. All the segments except the first and the last segments are complete, in the sense that the duration between control switches can be directly computed from the necessary conditions and the value of the Hamiltonian.

Denote the angular velocity and time for each segment by ω_i and $t_i, i = 1, 2, \dots, n$. Assume initial and goal orientation are θ_s and θ_g . We have:

$$\begin{aligned} \sum_{i=1}^n t_i \omega_i + \theta_s &= \theta_g \\ t_{\text{total}} &= \sum_{i=1}^n t_i. \end{aligned} \quad (4)$$

In equation ??, the equal sign means the same arc length on the circle. Since both initial and goal orientation are in $[-\pi, \pi]$, we can construct a function $S_\theta(\theta_s, \theta_g)$ that calculates the total change of orientation through the trajectory; denote $\sum_{i=2}^{n-1} t_i \omega_i$ by θ_{inner} , $\sum_{i=2}^{n-1} t_i$ as t_{inner} . Then,

$$\begin{aligned} t_1 \omega_1 + \theta_{\text{inner}} + t_n \omega_n &= S_\theta(\theta_s, \theta_g) \\ t_1 \omega_1 + t_n \omega_n &= S_\theta(\theta_s, \theta_g) - \theta_{\text{inner}} \end{aligned} \quad (5)$$

$$t_{\text{total}} = t_1 + t_{\text{inner}} + t_n. \quad (6)$$

B. Algorithm for each case

1) *Roll: spin-spin:* Assume a trajectory is a roll, and the first and last controls are both spins (take figure ?? as an example). In this case, the control line should be parallel to the line passing all the spin centers C_s , in which one is the first and the other is the last (order is not important). Denote the distance between first and last rotation centers by Δx .

Denote the segment length by Δl ; the orientation change for each segment by $\Delta \alpha$. Δl and $\Delta \alpha$ of roll in the control line can be calculated as follows:

$$\begin{aligned} \Delta l_{\text{spin}} &= 0 \\ \Delta l_{\text{turn}} &= 2\sqrt{r^2 - 4H^2} \\ \Delta \alpha_{\text{spin}} &= 2\pi/3 - 2\text{acos}(2H/r) \\ \Delta \alpha_{\text{turn}} &= 2\text{acos}(2H/r), \end{aligned}$$

since the distances from C_1 and C_s to the control line are $3H$ and H , which can easily be derived from the fact that Pontryagin's Principle indicates that H must be constant over a trajectory.

Recall that there are n segments, and let $k = \lfloor n/2 \rfloor$ be the number of turns between the first and last control. Therefore,

$$\begin{aligned} \Delta x &= k \times \Delta l_{\text{turn}} = 2k\sqrt{r^2 - 4H^2} \\ H &= (1/2)\sqrt{r^2 - \Delta x^2/4k^2}. \end{aligned} \quad (7)$$

The Hamiltonian value H can be calculated by looping over integer values of k , which is less or equal to 3, since optimal roll trajectory cannot contain more than six switches. If $H \in (2/\sqrt{3}, 2]$, we can then calculate the time for this roll trajectory by using equation ??, with $\omega_1 = \omega_2$ in this case.

$$\theta_{\text{inner}} = k * \Delta \alpha_{\text{turn}} + (k - 1) * \Delta \alpha_{\text{spin}} \quad (8)$$

$$t_{\text{inner}} = 3k * \Delta \alpha_{\text{turn}} + (k - 1) * \Delta \alpha_{\text{spin}}. \quad (9)$$

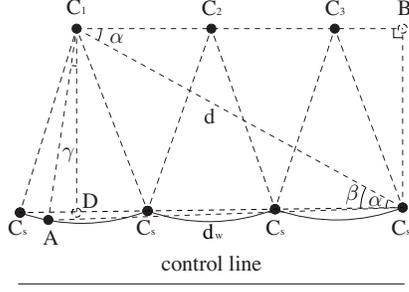


Fig. 6: A simplified case of a roll trajectory with different types of first and last controls

2) *Roll: turn-turn*: Assume the trajectory is roll, and both first and last control are turns (take figure ?? as an example). For this case, we consider rotation centers C_1 and C_3 . Denote $\|C_1 - C_3\| = \Delta c$. Also, $\omega_1 = \omega_n$, using equation ?? to solve for the time.

$$\begin{aligned} \Delta c &= k \times \Delta l_{\text{turn}} = 2k\sqrt{r^2 - 4H^2} \\ H &= (1/2)\sqrt{r^2 - \Delta c^2/4k^2} \end{aligned} \quad (10)$$

$$\begin{aligned} \theta_{\text{inner}} &= k\Delta\alpha_{\text{spin}} + (k-1)\Delta\alpha_{\text{turn}} \\ t_{\text{inner}} &= k\Delta\alpha_{\text{spin}} + 3(k-1)\Delta\alpha_{\text{turn}} \end{aligned} \quad (11)$$

$$t_1 + t_n = (S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}})/\omega_1. \quad (12)$$

3) *Roll: turn-spin*: Assume the trajectory is a roll, and the first and last controls are different; one is a turn and the other is a spin. Take figure ?? as an example, with start at A and goal at the rightmost C_s . (The geometry also works if the positions are reversed, with A the goal.)

Define $d = \|C_1 - C_s\|$, and define $d_w = \|A - C_s\|$. For n total segments, there are $k = \lfloor n/2 \rfloor - 1$ spins between the first and last control. In figure ??, $\|C_1 - C_2\| = \Delta l_{\text{turn}} = 2\|C_3 - B\|$.

$$\begin{aligned} \sqrt{d^2 - 4H^2} &= k \times \Delta l_{\text{turn}} = (2k+1)\sqrt{r^2 - 4H^2} \\ H &= \sqrt{((2k+1)^2 r^2 - d^2)/(2(2k+1)^2 - 4)} \end{aligned} \quad (13)$$

$$\begin{aligned} \theta_{\text{inner}} &= k\Delta\alpha_{\text{spin}} + k\Delta\alpha_{\text{turn}} = 2k\pi/3 \\ t_{\text{inner}} &= k\Delta\alpha_{\text{spin}} + 3k\Delta\alpha_{\text{turn}} \\ t_1\omega_1 + t_n\omega_n &= S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}}. \end{aligned} \quad (14)$$

To calculate the time, since $\omega_1 \neq \omega_n$, we need to calculate the time for the turn. Denote the angle $\angle AC_s C_1$ by β , the angle $\angle C_s C_s C_1$ by α , the angle $\angle AC_1 D$ by γ . If the projection of d_w onto the control line is longer than $\|C_1 - B\|$, then the first control is longer than half of a complete turn and vice versa. Therefore, the angle for the first control is $\Delta\alpha_{\text{rotate}}/2 \pm \gamma$. Use equation ?? to find the total time.

$$\begin{aligned} \alpha &= \text{asin}(2H/d) \\ \beta &= \text{acos}((d_w^2 + d^2 - r^2)/(2 \times d_w \times d)) \\ \Delta x &= d_w \cos(\beta - \alpha) - (k+1/2)\Delta l_{\text{turn}} \\ \text{sign} &= \Delta x/|\Delta x| \\ \gamma &= \text{acos}((2H + d_w \sin(\beta - \alpha))/r) \\ t_1 &= \frac{1}{2}\alpha_{\text{turn}} + \text{sign} \times \gamma \\ t_n &= (S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}} - t_1\omega_1)/\omega_n. \end{aligned} \quad (15)$$

$$t_n = (S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}} - t_1\omega_1)/\omega_n. \quad (16)$$

4) *Shuffle: spin-spin*: Calculating the value of Hamiltonian for a shuffle trajectory is almost the same as for the roll, except the segment lengths are different. Consider an example shuffle trajectory in figure ?? that starts with a spin at C_s on the left and ends with spin at C_s on the right. At P_1 , the control switches directly from C_1 to C_2 , so $\|C_1 - C_2\| = \sqrt{3}R$ and $\|C_1 - A\| = 6H$. Denote the distance between start and goal by d_w . We know from the structure of the shuffle trajectory that $\omega_1 = \omega_n$, and the sign of ω_3 (the angular velocity corresponding to the turn around C_2) is different from all the other angular velocities. Then, we have:

$$\begin{aligned} \Delta\alpha_{\text{spin}} &= 2\pi/3 - 2\text{acos}(2H/r) \\ \Delta\alpha_{\text{turn1}} &= \text{acos}(2H/r) - \pi/6 - \text{acos}(6H/\sqrt{3}r) \\ \Delta\alpha_{\text{turn2}} &= 2(\pi/6 - \text{acos}(6H/\sqrt{3}r)) \\ d_w &= 2\sqrt{r^2 - 4H^2} - 2\sqrt{3r^2 - 36H^2} \end{aligned} \quad (17)$$

$$\begin{aligned} \theta_{\text{inner}} &= 2 \times 3\Delta\alpha_{\text{turn1}}\omega_2 + 3\Delta\alpha_{\text{turn2}}\omega_3 \\ t_{\text{inner}} &= 3 \times 2\Delta\alpha_{\text{turn1}} + 3\Delta\alpha_{\text{turn2}} \end{aligned} \quad (18)$$

$$t_1 + t_n = (S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}})/\omega_1. \quad (19)$$

5) *Shuffle: turn-turn*: Assume the trajectory is shuffle and both the first and last controls are turns (take figure ?? as an example). There are two cases in this type of trajectory.

Case 1: The first and last turns have the same sign of angular velocity. Consider the case where the first control is C_1 and the last control is C_3 (the geometry also works with the reverse order, and gives the same time cost). Denote the distance between the first and last rotation center by Δc . Then $\|C_1 - C_3\| = \Delta c$, $\omega_1 = \omega_n$, and the sign of ω_2 is different from all the other angular velocities. Use equation ?? to calculate the total time:

$$\begin{aligned} c_w &= 2\sqrt{3r^2 - 36H^2} \\ H &= (1/6)\sqrt{3r^2 - c^2/4} \end{aligned} \quad (20)$$

$$\begin{aligned} \theta_{\text{inner}} &= 3\Delta\alpha_{\text{turn2}}\omega_2 \\ t_{\text{inner}} &= 3\Delta\alpha_{\text{turn2}} \end{aligned} \quad (21)$$

$$t_1 + t_n = (S_\theta(\theta_i, \theta_g))/\omega_1. \quad (22)$$

Case 2: The first and last turns have different signs of angular velocity. First, consider the case where the control sequence is from C_1 to C_2 (the control sequence from C_2 to C_3 is symmetric). Denote the distance between the two rotation centers by c_w . For the sequence from C_1 to C_2 , $\Delta c = \sqrt{3}r$. Assume the start is on the arc $C_s P_1$, denoted by P , let $\|C_3 - P\| = d_1$, and use equations ?? to calculate total time.

$$\Delta c = \sqrt{3}r \quad (23)$$

$$t_1\omega_1 + t_2\omega_2 = S_\theta(\theta_i, \theta_g) \quad (24)$$

$$t_1 = \text{acos}((r^2 + 3r^2 - d_1^2)/(2 \times r \times \sqrt{3}r)) \quad (25)$$

(In this case, the Hamiltonian value is not needed.)

Now consider the control sequence from C_2 to C_4 . Define $\Delta c = \|C_2 - C_4\|$, $d_2 = \|C_2 - C_s\|$. Assume the goal is on arc $C_s S_3$ and denoted by B . Define $d_3 = \|C_2 - B\|$, use equations ?? to calculate total time.

$$\sqrt{\Delta c^2 - 36H^2} = 2\sqrt{r^2 - 4H^2} - \sqrt{3r^2 - 36H^2} \quad (26)$$

$$\begin{aligned} \sqrt{d_2 - 16H^2} &= \sqrt{r^2 - 4H^2} - \sqrt{3r^2 - 36H^2} \\ \theta_{\text{inner}} &= 3\alpha_{\text{turn1}}\omega_2 + \alpha_{\text{spin}}\omega_3 \\ t_{\text{inner}} &= 3\alpha_{\text{turn1}} + \alpha_{\text{spin}} \end{aligned} \quad (27)$$

$$\begin{aligned} t_1\omega_1 + t_n\omega_n &= S_\theta(\theta_i, \theta_g) - \theta_{\text{inner}} \\ t_1 &= [\text{acos}((r^2 + \Delta c^2 - d_2^2)/(2 \times r \times \Delta c)) \\ &\quad - \text{acos}((r^2 + \Delta c^2 - d_3^2)/(2 \times r \times \Delta c))]/\omega_1 \end{aligned} \quad (28)$$

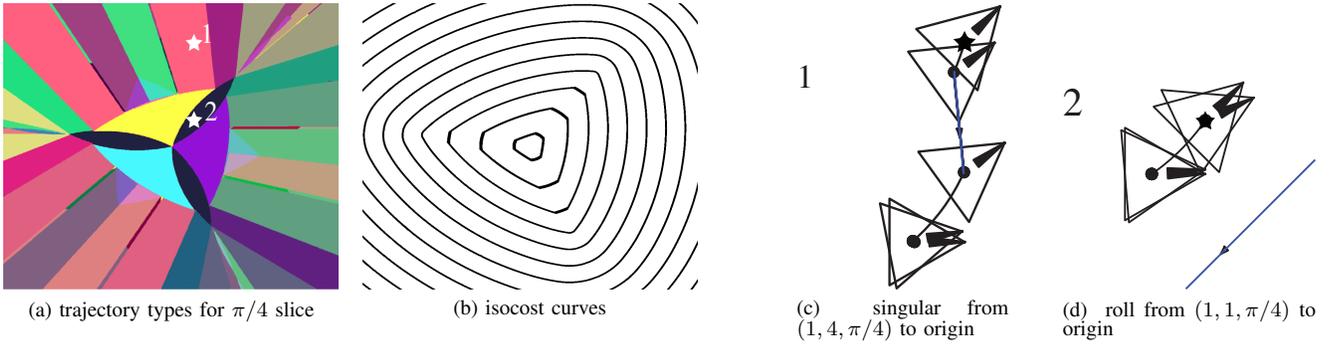


Fig. 7: Trajectories for $\theta_s = \pi/4$

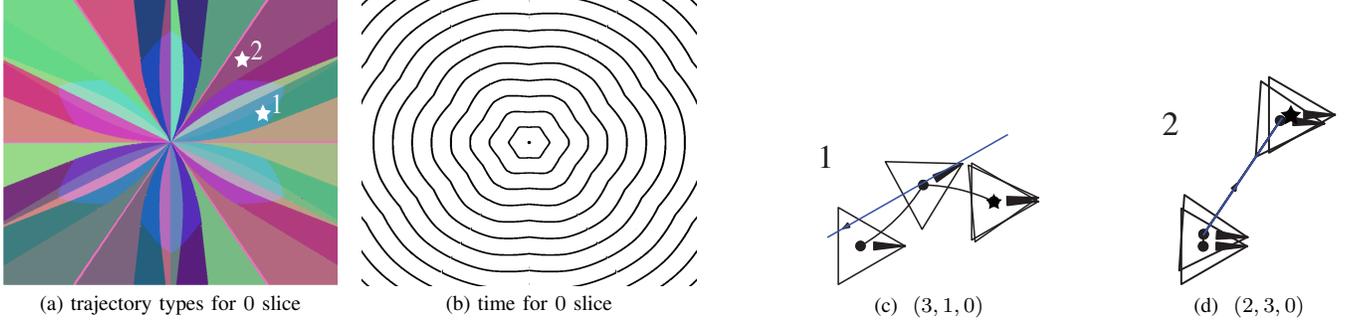


Fig. 8: Trajectories for $\theta_s = 0$

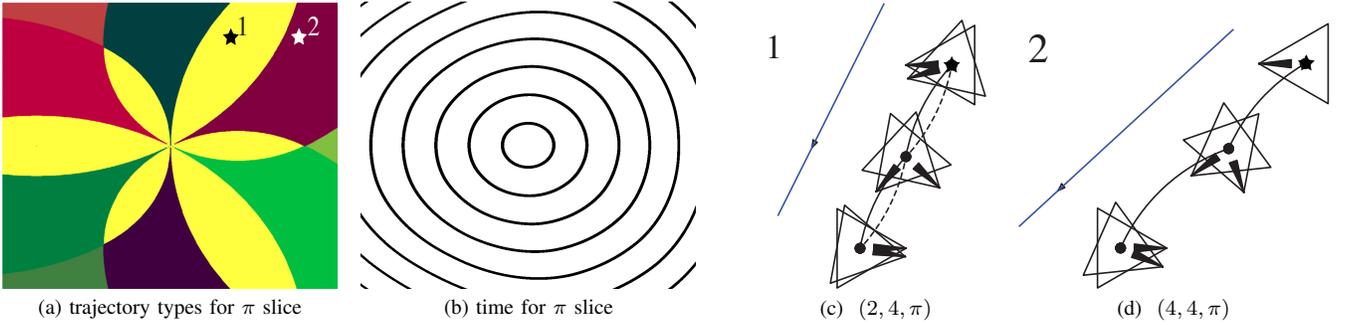


Fig. 9: Trajectories for $\theta_s = \pi$

6) *Shuffle: turn-spin: Case 1:* Initial and final turn and spin have same sign of angular velocity. Take figure ?? as example. Consider the control sequence from C_1 to C_3 , denote the distance between them by Δc . Assume initial A is on arc $C_s P_1$, Define $d_1 = \|A - C_2\|$. Use equations ?? to calculate total time.

$$\sqrt{\Delta^2 c - 4H^2} = 2\sqrt{3r^2 - 36H^2} - \sqrt{r^2 - 4H^2} \quad (29)$$

$$\begin{aligned} \theta_{\text{inner}} &= 3\Delta\alpha_{\text{turn1}}\omega_2 + 3\Delta\alpha_{\text{turn2}}\omega_3 \\ t_{\text{inner}} &= 3\Delta\alpha_{\text{turn1}} + 3\Delta\alpha_{\text{turn2}} \end{aligned} \quad (30)$$

$$\begin{aligned} t_1\omega_1 + t_n\omega_n &= S_\theta(\theta_i, \theta_g) \\ t_1 &= (\text{acos}(\frac{r^2 + 3r^2 - d_1^2}{2 \times r \times \sqrt{3r}}) - (\pi/6))/\omega_1 \end{aligned} \quad (31)$$

Case 2: Initial and final turn and spin have different sign of angular velocity. Take figure ?? as example, consider the case that control sequence from C_2 to C_s . Define $\Delta c = \|C_2 - C_s\|$, assume initial in on arc $P_1 P_2$ denoted by A , define $d_1 = \|C_s - A\|$. Use equations ?? to calculate total time.

$$\sqrt{\Delta c^2 - 16H^2} = \sqrt{r^2 - 4H^2} - \sqrt{3r^2 - 36H^2} \quad (32)$$

$$\begin{aligned} \theta_{\text{inner}} &= 3\Delta\alpha_{\text{turn1}}\omega_2 \\ t_{\text{inner}} &= 3\Delta\alpha_{\text{turn1}} \\ d_2 &= 2r \sin(\Delta\alpha/2) \\ t_1 &= [\text{acos}((r^2 + \Delta c^2 - d_1^2)/(2 \times r \times \Delta c)) \\ &\quad - \text{acos}((r^2 + \Delta c^2 - d_2^2)/(2 \times r \times \Delta c))]/\omega_1 \end{aligned} \quad (33)$$

IV. SIMULATION, EXPERIMENTS, AND COMPARISON

We implemented this algorithm in C. The algorithm is sufficiently fast (about 70 trajectories per second on a standard desktop), to allow dense sampling of the configuration space to explore how the control sequences differ for different initial configurations of the robot.

Figures ?? to ?? show the results for three different starting orientations of the robot, with the goal at the origin. For each configuration, we generate the best singular trajectory if it exists, and the best generic trajectory, and compare the two to find the optimal. In each figure, each different color represents a different control sequence (identified by the type: singular, roll or shuffle, and different first and last control). The time contour plots shows isocost curves for

TABLE I: comparison of time optimal trajectory and turn - drive - turn

| distance to origin | better in percentage |
|--------------------|----------------------|
| 0 - 1 | 10.1697% |
| 1 - 2 | 17.9340% |
| 2 - 3 | 22.5781% |
| 3 - 4 | 23.6106% |
| 4 - 5 | 22.9535% |
| 5 - 6 | 21.8520% |
| 6 - 7 | 20.7536% |
| 7 - 8 | 19.7472% |
| 8 - 9 | 18.5558% |
| 9 - 10 | 17.1047% |

time costs in increments of 0.5 seconds for the corresponding slice.

The result shows that the time-optimal trajectories for an omni-directional vehicle are mostly roll and shuffle for initial configurations not far (combine the distance and the angle difference) from the goal, and singulars otherwise. For initial angle close to the goal angle, three of six fastest translation directions that may be close to starting angle may be used by time-optimal trajectories depending on starting location (figure ??). When the starting angle is very close to the goal orientation, all six fastest translation directions may be used (figure ??). And if the angle difference between start and goal are big enough, the angular cost dominates (figure ??). Figure ?? and ?? also show that there are equivalent trajectories for some configurations, because the goal orientation can be attained by rotating either in the positive or negative direction with the same time cost. An example is shown in figure ??.

For configurations that are far away from the goal, we observe that a simple control strategy, which is a special case of singular, sometimes turns out to be time-optimal: turn the fastest translation direction to face the goal; drive to the goal; turn to the correct angle. Under these circumstances, is it worthwhile to implement the complete algorithm described here? We compared the time-optimal trajectory with the simple turn-drive-turn strategy for some configurations close to the origin, as shown for different distances from the start to the goal in table ?. The time-optimal trajectories turn out to be faster by 10% to 20%.

At a high level, we classify trajectories as “roll”, “shuffle”, or “singular”, but we might also simply describe a trajectory type based on the sequence of controls that occur in the trajectory. Using this counting method, we found 459 types of trajectory that are optimal for at least some configurations. This number compares to 40 trajectory types for the differential- drive and 46 for Reed-Shepp car.

We may explicitly enumerate all 134 different types for trajectories with only one or two segments. For rolls and shuffles with more segments, because of the strong constraints on the control sequences, the exact number of types can be enumerated; the results are listed in table ?. The situation is more complicated for singulars. We sampled the space of starting configurations in an area near the origin and found 193 different types of singular trajectories occurred; this is only a lower bound.

V. CONCLUSION

We developed an analytical method to efficiently find the time-optimal trajectory between configurations of an omni-directional vehicle. We sampled the space of starting configurations and used the algorithm to explore the distribution of the trajectory structures over the configuration space. We also explicitly counted the trajectory types for rolls and

TABLE II: all types of trajectory for three or more segments

| segment length | number of singular types | number of roll types | number of shuffle types |
|----------------|--------------------------|----------------------|-------------------------|
| 3 | 24 | 12 | 24 |
| 4 | 33 | 12 | 24 |
| 5 | 47 | 12 | 24 |
| 6 | 51 | 12 | 0 |
| 7 | 20 | 12 | 0 |
| 8 | 18 | 0 | 0 |

shuffles, and used the sampling to find a lower bound on the singular trajectory types. We also compared the time-optimal trajectories with other driving methods, and it turns out the the time optimal trajectories are somewhat faster for start configurations that are not too close or too far away from the goal.

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