# Structural Optimization of 3D Masonry Buildings (Supplemental Material)

## A Matrix Structure

We detail the matrix equation for static equilibrium:

$$\mathbf{A}_{eq} \cdot \mathbf{f} + \mathbf{w} = \mathbf{0}$$

$\mathbf{w}_j$: 6×1 vector containing the 3D weight and net torque for block $j$. Typically the only non-zero element is the $z$-component of weight. For any external loads acting on block $j$, the force and torque contributions are added here.

$\mathbf{r}_k$: Contains the unknown force vectors $\mathbf{f}^i$, for vertices $i$ on interface $k$. $height(\mathbf{r}_k)$ is $4v_k$, where $v_k$ is the number of vertices on interface $k$ and each vertex contributes a 3D force with positive and negative parts for the axial forces.

$\mathbf{A}_{j,k}$: Submatrices $\mathbf{A}_{j,k}$ contain coefficients for net force and net torque contributions from interface $k$ acting on block $j$. Each $\mathbf{A}_{j,k}$ has dimension $6 \times height(\mathbf{r}_k)$. Rows 1-3 are coefficients for net force contributions in $x, y, z$ and rows 4-6 are coefficients for net torque contributions about the $x, y, z$ axes.

$$\mathbf{A}_{j,k}\mathbf{r}_k = \begin{bmatrix} \mathbf{F}_k & \mathbf{F}_k & \dots \\ \mathbf{T}_{i,j,k} & \mathbf{T}_{i+1,j,k} & \dots \end{bmatrix} \begin{bmatrix} \mathbf{f}^i \\ \mathbf{f}^{i+1} \\ \vdots \end{bmatrix}$$

$\mathbf{F}_k = [\hat{\mathbf{e}}_{n_k} \ \hat{\mathbf{e}}_{u_k} \ \hat{\mathbf{e}}_{v_k}]$ and $\mathbf{T}_{i,j,k} = [(\hat{\mathbf{e}}_{n_k} \times \mathbf{v}_{i,j}) \ (\hat{\mathbf{e}}_{u_k} \times \mathbf{v}_{i,j}) \ (\hat{\mathbf{e}}_{v_k} \times \mathbf{v}_{i,j})]$. Unit vectors $\hat{\mathbf{e}}_{n_k}$, $\hat{\mathbf{e}}_{u_k}$ and $\hat{\mathbf{e}}_{v_k}$ are the normal vector and friction basis vectors for face $k$ (see Figure 1). The subscript $x$ refers to the $x$-component of the vector.

The number of submatrices $\mathbf{A}_{j,k}$ in row $j$ of $\mathbf{A}_{eq}$ is equal to the number of neighbors incident on block $j$. There are two submatrices in each column $k$, since $\mathbf{r}_k$ represents the interaction between surfaces of two adjacent blocks.

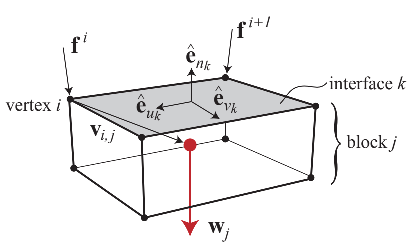


**Figure 1:** *Indexing for equations of static equilibrium. Vector $\hat{\mathbf{e}}_{n_k}$ is the unit normal for interface k, and $\hat{\mathbf{e}}_{u_k}$ and $\hat{\mathbf{e}}_{v_k}$ are the directions of in-plane friction forces. Vector $\mathbf{v}_{i,j}$ is the relative position of vertex i w.r.t. the centroid of block j. $\mathbf{w}_j$ is the 3D weight vector for block j.*

## B Partial Derivatives

The partial derivatives of coefficients for net force equilibrium, $\mathbf{F}_k$, on face $k$ are:

$$\frac{\partial \mathbf{F}_k}{\partial \omega} = \frac{\partial [\hat{\mathbf{e}}_n \ \hat{\mathbf{e}}_u \ \hat{\mathbf{e}}_v]_k}{\partial \omega}$$

where $\omega$ is a parameter from the set $\{u_{i,k}, v_{i,k}, n_k, \theta_k, \phi_k\}$ as described in §6. The partial derivatives of coefficients for net torque equilibrium, $\mathbf{T}_k$, on face $k$ are:

$$\begin{aligned} \frac{\partial \mathbf{T}_{i,j,k}}{\partial \omega} &= \frac{\partial [(\hat{\mathbf{e}}_n \times \mathbf{v}_{i,j}) \ (\hat{\mathbf{e}}_u \times \mathbf{v}_{i,j}) \ (\hat{\mathbf{e}}_v \times \mathbf{v}_{i,j})]_k}{\partial \omega} \\ \frac{\partial (\hat{\mathbf{e}}_n \times \mathbf{v}_{i,j})_k}{\partial \omega} &= \hat{\mathbf{e}}_n \times \frac{\partial \mathbf{v}_{i,j}}{\partial \omega} + \frac{\partial \hat{\mathbf{e}}_n}{\partial \omega} \times \mathbf{v}_{i,j} \\ &= \hat{\mathbf{e}}_n \times \left( \frac{\partial \mathbf{p}_{i,j}}{\partial \omega} - \frac{\partial \mathbf{c}_j}{\partial \omega} \right) + \frac{\partial \hat{\mathbf{e}}_n}{\partial \omega} \times \mathbf{v}_{i,j} \end{aligned}$$

The derivative of the centroid position $\mathbf{c}_j$ for block $j$ is:

$$\frac{\partial \mathbf{c}_j}{\partial \omega} = \frac{\partial}{\partial \omega} \left( \frac{\sum v_{T_{i,j}} \mathbf{c}_{T_{i,j}}}{\sum v_{T_{i,j}}} \right)$$

where $v_{T_{i,j}}$ is the volume of tetrahedron $i$ on block $j$ and $\mathbf{c}_{T_{i,j}}$ is the centroid of tetrahedron $i$.

$$\begin{aligned} \frac{\partial v_{T_{i,j}}}{\partial \omega} &= \frac{1}{6} sign(\mathbf{a}_0 \cdot (\mathbf{a}_1 \times \mathbf{a}_2)) \frac{\partial}{\partial \omega} (\mathbf{a}_0 \cdot (\mathbf{a}_1 \times \mathbf{a}_2)) \\ \frac{\partial \mathbf{c}_{T_{i,j}}}{\partial \omega} &= \frac{1}{4} \frac{\partial}{\partial \omega} (\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2) \end{aligned}$$

where coordinates $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ are three corners of tetrahedron $i$, offset such that the fourth coordinate lies at the origin $(0, 0, 0)$.

The derivative of the weight vector $\partial \mathbf{w} / \partial \omega$ for block $j$ is given by:

$$\frac{\partial \mathbf{w}_j}{\partial \omega} = \rho \frac{\partial v_j}{\partial \omega} \hat{\mathbf{g}} = \rho \left( \sum_i \frac{\partial v_{T_{i,j}}}{\partial \omega} \right) \hat{\mathbf{g}}$$

where $\rho$ is the block density and $\hat{\mathbf{g}}$ is the direction of gravity, and $v_j$ is the volume of block $j$.

### B.1 Constraint Derivatives

In the closed form solution of $\mathbf{f}^*$, the constraint matrix $\mathbf{C}$ is a concatenation of the matrix $\mathbf{A}_{eq}$ (static equilibrium), the active friction-cone inequalities of the matrix $\mathbf{A}_{fr}$, and the active lower bounds on $\mathbf{f}$. The friction constraints and lower bound constraints are not dependent on block geometry, giving $\partial \widetilde{\mathbf{A}}_{fr} / \partial \omega = \mathbf{0}$ and $\partial \widetilde{\mathbf{I}}_{lb} / \partial \omega = \mathbf{0}$.

$$\frac{\partial \mathbf{C}}{\partial \omega} = \begin{bmatrix} \partial \mathbf{A}_{eq} / \partial \omega \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \frac{\partial \mathbf{b}}{\partial \omega} = \begin{bmatrix} -\partial \mathbf{w} / \partial \omega \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

where $\omega$ is a parameterization of the structure's geometry $\Omega$. The derivations of $\partial \mathbf{A}_{eq} / \partial \omega$ and $\partial \mathbf{w} / \partial \omega$ are shown above.

### B.2 Energy Derivatives

The expression for the derivative of $\mathbf{f}^*$ is then obtained by differentiating the closed form expression:

$$\frac{\partial \mathbf{f}^*_\Omega}{\partial \omega} = \mathbf{H}^{-1} \left( \frac{\partial \mathbf{C}^T}{\partial \omega} \mathbf{E}^{-1} \mathbf{b} + \mathbf{C}^T \mathbf{E}^{-1} \left( \frac{\partial \mathbf{b}}{\partial \omega} - \frac{\partial \mathbf{E}}{\partial \omega} \mathbf{E}^{-1} \mathbf{b} \right) \right)$$

$\mathbf{H}$ is the weighting matrix for the objective function and is held constant, and $\mathbf{E} = \mathbf{C}\mathbf{H}^{-1}\mathbf{C}^T$. The derivative of $\mathbf{E}$ by application of the chain rule is:

$$\partial\mathbf{E}/\partial\omega = \partial\mathbf{C}/\partial\omega\, \mathbf{H}^{-1}\mathbf{C}^T + \mathbf{C}\, \mathbf{H}^{-1}\partial\mathbf{C}^T/\partial\omega$$

The terms $\partial\mathbf{C}/\partial\omega$ and $\partial\mathbf{b}/\partial\omega$ describe how the constraints change as the geometry changes according to parameterization $\omega$. The expression for the gradient of $y(\Omega)$ is given by:

$$\nabla y \quad = \quad \alpha\nabla y_{uniform} + \nabla y_{torque}$$

where the derivatives of uniform and torque tension energies are:

$$\begin{aligned} \frac{\partial y_{uniform}}{\partial\omega} &= \mathbf{f}^{*T}\mathbf{H}_{uniform}\frac{\partial\mathbf{f}^*}{\partial\omega} \\ \frac{\partial y_{torque}}{\partial\omega} &= \mathbf{f}^{*T}\mathbf{H}_{torque}\frac{\partial\mathbf{f}^*}{\partial\omega} + \frac{1}{2}\left(\mathbf{f}^{*T}\frac{\partial\mathbf{H}_{torque}}{\partial\omega}\mathbf{f}^*\right) \end{aligned}$$

with $\partial\mathbf{H}_{torque}/\partial\omega = (\mathbf{I} - \mathbf{H}_{min})^T\partial\mathbf{D}_{torque}/\partial\omega(\mathbf{I} - \mathbf{H}_{min})$. Matrices $\mathbf{H}_{uniform}$ and $\mathbf{H}_{min}$ are constant since it assumed minimum-tension vertices remain the same for differential movement.

## C Cables

For gradient computation, we parametrize cables using the x,y,z coordinates of their end points. The derivative of the weight vector $\partial\mathbf{w}/\partial\omega$ for cable is given by:

$$\frac{\partial\mathbf{w}}{\partial\omega} = \rho\frac{\partial L}{\partial\omega}\hat{\mathbf{g}}$$

where $L$ is the length of the cable, and $\rho$ is its the mass per unit length.

$$\frac{\partial L}{\partial\omega} \quad = \quad \frac{1}{2}\frac{(\mathbf{p}_0 - \mathbf{p}_1)\cdot\frac{\partial\mathbf{P}_0}{\partial\omega}}{\|\mathbf{p}_0 - \mathbf{p}_1\|}$$

where coordinates $\mathbf{p}_0, \mathbf{p}_1$ are two ends of the cable.

The derivative of the cable tension direction is

$$\begin{aligned} \frac{\partial\hat{\mathbf{e}_t}}{\partial\omega} &= \frac{\partial}{\partial\omega}\left(\frac{\mathbf{p}_0 - \mathbf{p}_1}{\|\mathbf{p}_0 - \mathbf{p}_1\|}\right) \\ &= \frac{\|\mathbf{p}_0 - \mathbf{p}_1\|\cdot(\frac{\partial\mathbf{p}_0}{\partial\omega}) - (\mathbf{p}_0 - \mathbf{p}_1)\cdot(\frac{\partial\|\mathbf{p}_0 - \mathbf{p}_1\|}{\partial\omega})}{\|\mathbf{p}_0 - \mathbf{p}_1\|^2} \end{aligned}$$