Structure and geometry of minimum-time trajectories for planar rigid bodies
(extended abstract)

Andrei A. Furtuna, Weifu Wang, Yu-Han Lyu, Devin Balkcom

Abstract—This paper explores the structure of the time optimal trajectories for a rigid body in the unobstructed plane. The available controls for the system are the velocity and angular velocity of the body, in its own frame of reference. Controls are bounded by a closed convex polyhedron. The paper shows that the structure of the optimal trajectories may be completely described analytically.

I. INTRODUCTION

Efficient motion is a core problem in robotics. What is the fastest way to move a simple mobile robot in the plane from one place to another? Even with a kinematic model, only a few simple controls, and without obstacles, nobody knows, except in a very few special cases (the most well-known of which are the steered cars first studied by Dubins [8] and by Reeds and Shepp [15]).

If the system of equations that describes the trajectories is complicated, or if the cost function is complicated, we do not expect there to be any hope of finding an exact analytical description of the optimal trajectories. Typical approaches to path-planning problems therefore discretize the state space and the control space, and search or use dynamic programming. The results are often crude. Strong and unprovable assumptions (for example, sharply limiting the number of control actions, or ignoring many of the available controls) made for computational reasons can lead to trajectories that are incorrect or unnecessarily complicated.

The work outlined in this paper attempts to bridge the gap between strong analytical results for simple robot systems, and general-purpose planning algorithms. We present a general solution to the problem of finding minimum time trajectories for a self-propelled, velocity-bounded rigid body in the obstacle-free plane. (By self-propelled, we mean that constant constraints on the velocity are described in the frame of the robot, rather than in the world frame. Thus, a steered car might have a maximum ‘forwards’ speed; the ‘forwards’ direction is determined by the heading of the robot.)

Many of the results presented in this paper were previously presented (in much greater detail) by the Ph.D. thesis of Andrei Furtuna [9]; this paper merely attempts to summarize key results, with a focus on the geometry and topology of the space of optimal trajectories for planar rigid bodies. Section V will present new work on solving a somewhat related problem, in which the available controls are chosen to be a discrete set of points in the control space, and a cost is charged for switching between these controls.

Formally, let the state of the body be given by $q = (x, y, \theta)$, the location and orientation of some frame rigidly attached to the body. The accelerations are assumed to be instantaneous, allowing direct control over the velocity. The controls are given by $u \in \mathbb{R}^3$; the first two elements of $u$ represent the translational velocity of the robot measured relative to the body-attached frame, and the third element represents the angular velocity of the frame. For example, if we attach a frame to a steered car such that the first axis of the frame points along the forward driving direction the car, the control $u = (1, 0, -1)$ would correspond to driving forwards with speed 1, while turning to the right with unit angular velocity.

Given a Lebesgue-integrable control function $u(\cdot)$, a trajectory is a parameterized curve:

$$q(t) = q_0 + \int_0^t R(\theta(\tau))u(\tau) \, d\tau$$

where

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ (2)

Given a start configuration $q_0$, a goal configuration $q_g$, and a particular set of linear constraints describing a closed, convex polyhedron $U \subset \mathbb{R}^3$ such that $u(t) \in U$, the problem is to find a control trajectory $u(t)$ and a final time $T$ such that $q(T) = q_g$ and $T$ is minimized over all admissible control trajectories.
This problem generalizes several previously studied problems in optimal control. For example, the fastest trajectories for a steered car that can only drive forwards, with maximum velocity of 1 and bounded steering angle were found by Dubins [8] in the context of studying shortest curvature-constrained paths in the plane: \( U = [1, 1] \times [0, 0] \times [-1, -1] \) for the Dubins car. As a second example, a model of a 'differential-drive' robot with two powered wheels, with independent bounds on the velocities of the wheels, has a diamond-shaped set of constraints on \( U \) corresponding to the tradeoff between angular and translational velocities induced by wheel-speed constraints.

Section II will show that, regardless of the shape of \( U \), Pontryagin’s Maximum Principle (PMP) [14] imposes certain necessary conditions on any time-optimal trajectory of a self-propelled rigid body described by the control equations above. These necessary conditions have a very simple geometric interpretation: except for a few special cases that we will describe later, every optimal trajectory maximizes the speed of some bead that is being pushed along some line in the plane. The bead is located at the point on the line closest to the base of the body-attached frame.

Given a start configuration, the goal configuration, and the particular control constraints \( U \), the problem may then be phrased as finding the location of this control line in the plane such that if maximizing controls are applied at each instant, the body is brought as quickly as possible to the goal configuration. For certain specific, simple systems, this problem can be solved exactly and analytically.

In order to study the types of trajectories that might arise for more complicated systems (for example, a car that turns more quickly to the left than the right, or a polygonal rigid body being pushed stably by a mobile robot), we consider a related problem. Given a placement of the rigid body relative to the control line and a set of constraints \( U \), what trajectory is generated by the control line? It turns out that piece-wise constant controls that only make use of the vertices of \( U \) and particular translations (essentially, bang-bang controls) are sufficient to generate optimal trajectories; we will say that the sequence of vertices applied during a trajectory is the ‘structure’ of that trajectory.

By considering all possible placements of a rigid body with respect to a line, we may enumerate trajectories that arise. An interesting connection to topology may be drawn here. We show that each trajectory may be characterized by a particular constant value (the Hamiltonian). Considering level sets of the Hamiltonian function shows that there are finitely many singular values of the Hamiltonian, such that trajectory structure only changes at these easily-computable singular values. We may therefore use these critical values to algorithmically completely enumerate all possible trajectory structures for a particular system, and in fact, the number of trajectory structures is only polynomial in the number of vertices of \( U \). This exact enumeration of trajectory structures allows robust numerical techniques that quickly and provably find time-optimal trajectories for any system of the class we consider.

A. Related work

We trace the roots of the current inquiry to a problem posed by Markov in 1887 ([13]): what are the shortest planar curves, of bounded maximum curvature and connecting two given tangent vectors? Seventy years later, the solution to this problem was characterized by L. E. Dubins ([8]), who described a class of curves he called R-geodesics. In 1975, Cockayne and Hall ([6]) showed how to synthesize the shortest R-geodesics to any given configuration.

In 1990, Reeds and Shepp ([15]), explicitly motivated by a robotics problem (finding the optimal trajectories for a robotic cart), and by Dubins’ success at giving, in effect, the shortest paths for a car that can only travel forwards, gave a characterization of the geodesics for a car that can travel both forwards and backwards. Their result revealed the scope for the application of optimal control theory into mobile robotics, and shortly afterwards two independent papers (by Sussmann and Tang [19] in 1991 and Boissonnat et al. [4] in 1992, respectively) re-established (and even slightly tightened) the Reeds and Shepp results on the more general basis of classical optimal control theory. The optimal trajectory synthesis for the Reeds-Shepp car was given by Souères and Laumond in 1996 [17]. The Pontryagin Principle constituted the main theoretical basis of these papers, and at this point we see coming together the approach of which the present work is the logical continuation: analyzing planar motion problems through the prism of the Pontryagin Principle.

A time-optimal model of the differential drive was studied by Balkcom and Mason in 2002 [3]. They obtained a novel set of curves, as well as an optimal trajectory synthesis for the differential drive, by considering the controls to be wheel velocities rather than accelerations. There is a good case to be made that such a “kinematic” (as opposed to dynamic) model is not only convenient to analyze, but also reasonable. Common electric motors respond to a given voltage by quickly settling to a well-determined velocity. The input can thus be considered to be, in effect, this velocity; and the time to cover a given distance \( d \) under a given control is better estimated through dividing the distance by the “steady” velocity corresponding to the control, rather than through the \( \frac{1}{2} ma^2 \) formula.

The optimal control problem for dynamic models appears to be very difficult, as the differential equations describing the trajectories do not have recognizable analytic solutions, and, in some cases, the optimal trajectories appear to involve chattering (“the Fuller phenomenon”), an infinite number of control switches within a finite time. Such issues occurred in analyses of the bounded acceleration Dubins car (studied by Sussmann in 1997 [18] and by Souères and Boissonnat in 1998 [16]) and of a dynamic model of an underwater vehicle (studied by Chyba and Haberkorn in 2005 [5]).

On the other hand, the combination of kinematic models and the Pontryagin Principle seems more amenable to analytic solution. Such a model was used by Balkcom et al. in 2006 to characterize the time optimal paths for a symmetric omnidirectional vehicle [2] and by Furtuna and Balkcom in
II. Pontryagin’s Maximum Principle

Pontryagin’s Maximum Principle [14] places necessary conditions on the structure of optimal trajectories. Application of the Maximum Principle to the system equations, together with integration of the resulting differential equations, yields theorem 1 from [9]:

**Theorem 1:** Consider a rigid body in the unobstructed plane, with configuration \( q = (x, y, \theta) \), controls \( u \) in some set of constant vectors, and system equations

\[
\dot{q} = R(\theta)u, \tag{3}
\]

For any time optimal trajectory of the system, there exist constants \( k_1, k_2, \) and \( k_3 \), not all 0, such that at every time, the controls maximize the Hamiltonian equation:

\[
H = k_1 \dot{x} + k_2 \dot{y} + \dot{\theta}(k_1 y - k_2 x + k_3). \tag{4}
\]

Furthermore, the Hamiltonian is constant and positive over the trajectory.

III. Taxonomy of Optimal Trajectories

A remarkably rich set of trajectory types and structures arises from the simple-seeming equation 4. We present only a brief survey; [9] gives a detailed analysis of each. The basic approach to classifying trajectories is to consider possible values for the constants \( k_1, k_2, \) and \( k_3 \).

A. Whirl trajectories

Consider first the case when \( k_1 = k_2 = 0 \). In this case, Equation 4 reduces to:

\[
H = \dot{\theta}k_3 \tag{5}
\]

In words, this case tells us that trajectories that maximize (or minimize, if \( k_3 \) is negative) angular velocity at almost every time may be time optimal. For some systems, this result is quite simple. For example, a differential-drive robot with uniform independent bounds on wheel speeds maximizes angular velocity when the right wheel drives forwards at full speed, and the left wheel drives in full reverse; the vehicle spins in place counter-clockwise. It is not surprising that the optimal trajectory to change the angle of a differential-drive involves spinning in place.

For systems with many different controls that maximize angular velocity, this case is more complicated; the Maximum Principle seems to tell us very little. Indeed, for such systems, there may be infinitely many optimal trajectories that connect start and goal. Consider the Reeds-Shepp car, which can drive forwards or backwards. To change the orientation of the vehicle without (ultimately) changing the location, we might use a “three-point turn” – drive forwards and backwards while maintaining a constant angular velocity. But we would also expect a five-point or seven point turn to take the same amount of time, for the infinite-acceleration model considered.

Fortunately, whenever one optimal ‘whirl’ trajectory (constant angular velocity) exists, there always exists an optimal trajectory of a simple class we call roll-and-catch, and this trajectory may be found analytically. Although we omit the proof of the optimality of these trajectories and the technique for determining the particular trajectory (these details may be found in [9]), the basic structure is simple, and is shown in Figure 2. Find the convex hull of the rotation centers used along the trajectory. Then rotate around this point until the rigid body reaches the goal configuration.

B. Control-line trajectories

Having dealt with whirl trajectories, we turn to the more complex case for which the magnitude of the vector \( (k_1, k_2) \) is non-zero. Without loss of generality, we may consider \( k_1^2 + k_2^2 = 1 \).

There is now an interesting geometric interpretation of Equation 4. The term \( k_1 \dot{x} + k_2 \dot{y} \) is the projection of the linear velocity of the rigid body onto some vector \( k_1, k_2 \). The term \( \dot{\theta}(k_1 y - k_2 x + k_3) \) is the distance of the rigid body \( (x, y) \) from some line defined by the constants \( (k_1, k_2, k_3) \), scaled by \( \dot{\theta} \) (see figure 3).

Thus, given a particular \( (k_1, k_2, k_3) \) value, we can draw a directed line (specified by \( k_1 y - k_2 x + k_3 = 0 \)) in the plane,
Rigid body

Control Point

Fig. 3: A rigid body instantaneously following a control line optimal trajectory. Two possible reference points attached to the rigid body are shown.

Fig. 4: The control line uniquely determines a section of an optimal trajectory for a Dubins car.

such that any optimal trajectory must maximize the sum of its projected linear velocity and its distance-scaled angular velocity relative to this line.

Consider, for example, a case where the rigid body is initially close to the line, and pointing roughly in the direction of the line. The body might initially maximize the Hamiltonian using a translation control, assuming such a control is available. As the body translates, it may get far enough from the line that some rotation control is available that maximizes the Hamiltonian; at this point, the trajectory switches to the new control. In this fashion, the trajectory ‘bounces’ along the control line until the goal is reached.

There are three classes of control line trajectories: generic, singular, and tacking.

1) Generic trajectories: Generic trajectories are those for which the initial placement of the rigid body with respect to the control line completely determines the structure of the optimal trajectories. Along a generic trajectory, the set of times for which multiple controls maximize the Hamiltonian is of measure zero; these switching times correspond to well-defined switches between controls. Figure 4 shows an example of a generic trajectory for the Dubins car.

2) Singular trajectories: Translation of the rigid body parallel to the control line does not change the set of currently maximizing controls, since the maximizing controls depend on the distance and angle of the body relative to the control line. Therefore, such translation may occur indefinitely, and be followed by any of several other maximizing controls. Therefore, in this singular case, no single trajectory structure is determined by the configuration of the body with respect to the control line; there may be many trajectories consistent with the configuration, all of which satisfy the Maximum Principle. Figure 5 shows a case where such a singular trajectory is optimal, for a particular model of a mobile robot.

3) Tacking trajectories: The remaining type of trajectory that must be considered occurs in this case when multiple translation directions simultaneously maximize the Hamiltonian. Since translation along one of these directions will typically leave the other translation directions maximizing, there may be a family of tacking trajectories that all satisfy the Maximum Principle. Fortunately, since translation is commutative, rapid switching between a pair of translation directions need not be considered, and it is relatively straightforward to analyze these trajectories as a simple extension of generic or singular trajectories.

IV. TOPOLOGY OF SWITCHING SPACE

The taxonomy of trajectory structures in the previous section hints at an approach to determining the time-optimal trajectory between a pair of configurations: somehow enumerate all singular and generic trajectory types (sequences of constant controls) for a particular $U$, somehow choose an optimum trajectory from each type that reaches the goal, and then finally compare these candidates for optimality. Finally return the structure and duration of each control.

However, without more knowledge of the space of extremal trajectories (those satisfying the Maximum Principle), we might fear that there are very very many trajectory structures to consider (perhaps exponential in the number of constant controls that may appear in optimal trajectories). Fortunately, this turns out not to be the case. The space of extremal trajectories can be compactly represented using a structure we call the switching space, and analysis of the topology of this space shows that 1) the number of types of trajectory structures is actually fairly small, and 2) a useful upper bound on the number of segments in an optimal trajectory can be computed.

Given some available controls $U$, the structure of a generic trajectory is entirely determined by the distance of the rigid body from to the control line, and the orientation of the body.
Since application of a constant rotation control along a generic trajectory changes the relationship between the distance of the body from the control line and the angle of the body in a sinusoidal fashion, level sets of the Hamiltonian correspond to simple spins in place. Below these, labelled as ‘Roll’ trajectories, we find curves that alternate horizontal segments and sinusoidal segments; these curves wrap around the cylindrical switching space in the $\theta$ direction. Then there are curves that do not wrap around the entire cylinder, the ‘Shuffle’ trajectories.

Geometrically, we can see that there are different types of level curves that arise in Figure 6. The simplest are the straight horizontal lines at the top and bottom of the figure corresponding to simple spins in place. Below these, labelled as ‘Roll’ trajectories, we find curves that alternate horizontal segments and sinusoidal segments; these curves wrap around the cylindrical switching space in the $\theta$ direction. Then there are curves that do not wrap around the entire cylinder, the ‘Shuffle’ trajectories.

These trajectory types for this system geometrically correspond to ranges of values of the Hamiltonian, and are separated by certain critical curves, shown in black. We can in fact show that:

Lemma 1 (Lemma 15 from [9]): There exists a partitioning of the values of the Hamiltonian into a finite set of open intervals and a finite set of critical values, such that every canonical trajectory with a Hamiltonian within a single interval, containing the same control switch, will contain the same sequence of control switches, following the switch that the trajectories have in common.

The previous lemma suggests an algorithm for enumerating all generic trajectory structures. First, enumerate all critical values of the Hamiltonian. (Critical values of the Hamiltonian correspond to singular trajectories, shown as thick curves in Figure 6.) Then choose a set of values for the Hamiltonian: one value in between each pair of critical values, one value greater than the maximum critical value, and one value below the minimum critical value. For each value, enumerate all possible control switches (one possible switch corresponds to each possible pair of controls that may occur in sequence in a trajectory); if there are $n$ potential controls, then there will be no more than $n(n-1)$ such pairs. For each pair, compute the time to the next switch using the geometry of the control line, and repeat, ‘simulating’ the trajectory (this simulation can be done analytically and exactly). Lemma 15 suggests that the first switch between a particular pair of controls will repeat, so generic trajectory structures are periodic. Once one period of each trajectory type has been collected, the algorithm terminates.

A. Computing optimal trajectories between start and goal configurations

For some simple examples of rigid body systems, including the Dubins and Reeds-Shepp steered cars, differential drives [3], and a model of omnidirectional robots [20], the optimal trajectory classes can be enumerated and analyzed by hand, leading in each case to a complete algorithm that can be used to identify analytically the optimal trajectory between a given pair of configurations.

For more complex rigid-body systems, and in particular systems that have asymmetric control bounds, optimal trajectory types can still be enumerated analytically, but the inverse kinematics problem of solving for the location of the control line that causes the body to move from start to
(a) The Dubins car, a car that can only drive forwards.

(b) The Reeds-Shepp car.

(c) The differential-drive, a wheelchair-like vehicle.

Fig. 7: Switching spaces and example trajectories for standard robotic vehicles. For each vehicle, the figure on the left side shows level curves of the Hamiltonian in the $(y,\theta)$ space. The figure on the right side shows some extremal trajectories, in the plane, that correspond to portions of the Hamiltonian level curves on the left.
goal can only be solved analytically. Although we omit the details in this paper, it is possible to sample the space of line locations in such a way as to ensure that an approximately optimal trajectory can be found to a particular goal, with any desired degree of precision [21]. Since it turns out that the function describing the final location of the rigid body after some time is not Lipschitz continuous in the function describing the final location of the rigid body, any desired degree of precision [21]. Since it turns out that the function describing the final location of the rigid body after some time is not Lipschitz continuous in $k_1$, $k_2$, and $k_3$, this sampling must be done in a somewhat indirect way; the interested reader is referred to [21].

V. AN OPEN PROBLEM: OPTIMAL SEQUENCES OF MOTION PRIMITIVES

In this section, we describe an open problem that we believe to be very fundamental to understanding how analytical work on solving for optimal motion (like that described in this paper) may be extended to systems more complex than planar rigid bodies.

The approach described in this paper has limitations that appear to obstruct extension to more general systems:

1) Lack of geometric trajectories. Even with comparatively simple linear constraints on controls, trajectories may be followed that can only be described by differential equations, and optimal trajectories may be of this class for systems more complex than kinematic planar rigid bodies.

2) Chattering. Optimization criteria often leads to trajectories that are in some sense “bang-bang”, using a finite set of controls from a larger set. It may be the case that for any trajectory, there exists a trajectory with a smaller value of objective function that contains more switchings between controls.

On the other hand, most commonly used approaches to robot motion planning (see, for example, description of RRT or PRM algorithms in [12]) rely on discretizing the space of controls available to the system. Before applying such a planner, the first step is typically to choose such a space of motion primitives, and show that if the system has a connecting path between a pair of configurations, there is also a connecting path that makes use of only the selected motion primitives.

Here then is the simple problem statement: given a collection of available motion primitives, an objective function to minimize, and a pair of configurations, find a sequence of primitives, and corresponding durations, that minimize the objective function.

Using motion primitives works around the possibility that optimal trajectories might only be describable by differential equations; we only consider primitives that are geometric. A fascinating open question is what is lost by constraining the search in this way.

A deeper problem is that using motion primitives may only exacerbate the difficulty of chattering. However, we have shown that (not surprisingly) charging some fixed cost (a positive real number) for switching between controls guarantees that if a trajectory of bounded duration exists between a start and goal, then an optimal trajectory exists, if this switching cost is positive [9].

A. A simple example: the bench-mover’s problem

Consider the following simple example. A mover would like to move a heavy park bench. The mover is alone, so decides to lift one side of the bench, rotate the bench around the legs on the ground a bit, walk to the other side, lift the other side, rotate, and repeat. A simplified model represents the bench by a line segment, with available rotation centers on each endpoint. The mover can rotate the bench around either endpoint with angular velocity 1 either clockwise or counterclockwise. What is the fastest sequence of rotations to move the bench to a new location?

Unfortunately for the mover, some goal locations apparently close to the start will require running back and forth between the ends of the bench infinitely many times. Consider moving the line segment along the direction of a vector normal to the line segment. No matter how many rotations are in the trajectory, it is always “faster” to move the bench by making a sequence of more rotations, each with a shorter duration, since this allows a better approximation of a straight-line motion of the center of the bench. The simplest solution is to charge some small amount (perhaps 1 second) for walking between the ends of the bench.

B. Approach

We have made some initial progress on the study of optimal trajectories with costly switches, but only outline the approach and outline a few of the results here, as the work is still preliminary.

The most promising approach to this problem that we have tried is to separate the discrete problem of finding the correct structure (sequence of primitives) from the continuous problem of finding the correct durations of each primitives. For example, we might ask, “given the bench mover’s problem above, what is the fastest trajectory of type $R_+L^-R^L^-$?"
**Fig. 9:** Mixed trajectory for the bench mover’s problem with initial configuration \((-2.8, 3.05, \pi/4)\). First two controls have the same angular velocity and hence they are collinear and parallel to the control line. This is the optimal trajectory for this initial configuration with switching cost 1.

(The letters R and L describe which end to rotate about, and the signs indicate the direction of rotation.) If we could solve this problem for each possible structure, then we could compare across structures to find the minimum cost structure for a particular starting configuration.

**C. Early results**

We only sketch a few results of current work. For any particular trajectory structure of a rigid body in the plane with costly switches and a finite set of available translation and rotation controls, we find that there is a quantity very similar to the Hamiltonian for planar rigid bodies with linear velocity constraints and cost-free switches. The central equation is the same, and the value $H$ is still required to be constant, but the maximization condition from the Maximum Principle has disappeared:

**Theorem 2:** For a planar rigid body with a finite set of discrete controls, for any fixed control sequence $u_i$, any regular optimal duration $t^*$ in the costly switch model satisfies the following property: there exist constants $H > 0$, $k_1$, $k_2$, and $k_3$, such that for any control $u_i$ with the instantaneous velocity $(v_x, v_y, \omega)$ in the world frame when $u_i$ is applied at configuration $(x, y, \theta)$, we have

$$k_1 v_x + k_2 v_y + \omega (k_1 y - k_2 x + k_3) = H,$$

where $k_1^2 + k_2^2 \in \{0, 1\}$.

Further geometric analysis of each possible trajectory structure, using this necessary condition, shows that the bench mover’s problem has essentially three classes of optimal trajectories: *whirl*, *alternating* (strict alternation of rotation directions as rotation center shifts from left to right), and *mixed* which intersperse alternations with rotations through $\pi$ radians. We have also found a complete algorithm that finds an analytical solution to the problem of connecting a pair of configurations with an optimal trajectory, after a brief discrete search over possible trajectory structures.

**REFERENCES**


