

# Introducing robotic origami folding

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**Abstract**—Origami, the human art of paper sculpture, is a fresh challenge for the field of robotic manipulation, and provides a concrete example for many difficult and general manipulation problems. This paper presents some initial results, including the world’s first origami-folding robot, definition of a simple class of origami for which we have designed a complete automatic planner, an analysis of the kinematics of more complicated folds, and some new theorems about foldability.

## I. INTRODUCTION

The design and manipulation of foldable objects like paper bags, newspapers, fast-food containers, tents, mirrors for space telescopes, cardboard boxes, and clothing present some challenges; we hope that exploring origami folding will lead to some new insights into these hard problems. The primary contribution of this paper is the definition of a new task domain; although origami has been widely studied in the mathematics community, this appears to be the first examination of origami from the robotic manipulation perspective. The results presented fall in three classes: designing for foldability, folding manipulation, and analysis of closed chains.

There is a rich field of work on origami design in the mathematics community; Demaine *et al.* [1] provides a survey. Robert Lang’s papers and TreeMaker software [7] are usually credited with being the first in-depth work. Our contributions to origami design are the extension of some well-known necessary conditions for origami patterns to be flat-foldable. Work by Hull [5] provides a good survey, and is a starting point for our results.

Our primary interest in origami is manipulation. Much can be learned from human origami experts. One observation is that most origami folds are quite simple. Folds are introduced sequentially, and previously folded facets are ‘locked’ down, allowing the origami to be treated as a serial chain. We define two basic folding techniques, and present a complete planner for these types of folding. Our folds are similar to those used by the virtual origami software described in Miyazaki *et al.* [14]; our contribution relative to this work is a careful analysis of the conditions under which these folds can be made, a requirement for automatic planning. We present a paper airplane and a hat that are simply foldable, and show that the traditional ‘samurai hat’ design can be folded using eight simple folds and a single book fold, without flipping the paper. We present a prototype of a machine that can fold simply-foldable origami – a movie is available at <http://www.cs.cmu.edu/~devin>.

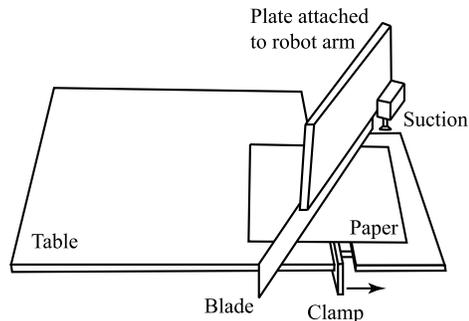


Fig. 1. A machine that can fold simple origami.

Box folding and sheet metal bending are perhaps the two applications in robotic manipulation closest to origami folding. Lu and Akella ([10],[11]) describe a planner (modelled on traditional planners for robot arms) that enumerates all collision-free sequences to fold a carton blank into a carton. Possibly the most complete and practical system for sheet metal bending is that described by Gupta *et al* [2], consisting of a high-level planner that determines the sequence of bends, and a number of low level planners for individual actions. Unlike sheet metal and cardboard cartons, origami is often flat when folded, and typically contains closed chains in its kinematic structure. These characteristics have implications for collision detection, state representation, and available actions.

Paper facets around a vertex form a spherical closed chain, and complicated folds like reverse folds and prayer folds manipulate the paper as a closed chain. Huffman [4] and McCarthy [12] both solve four-bar spherical linkages to show the relationship between joint angles; we extend this work to the case of n-bar linkages.

Although beyond the scope of this paper, planning for simultaneous manipulation of links in closed chains is a primary direction of future work; probabilistic methods of planning for closed chains are described in LaValle *et al.* [8] and Han and Amato [3]. Lenhart and Whitesides [9] and Milgram and Trinkle [13], [15] describe complete and efficient planners for the specific case of planar n-bar linkages without self-intersections.

## II. REPRESENTING ORIGAMI

Define the *facets* of an origami piece to be the unfolded regions of paper. In the present work, we model only polygonal facets, and treat them as rigid links. The facet edges interior to the paper are *creases*; a *crease line* is a set

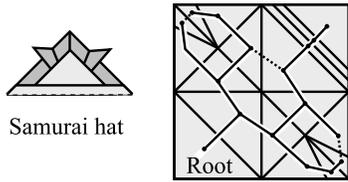


Fig. 2. The pattern, facet graph, and a facet tree for a samurai hat.

of colinear creases. Define the *origami pattern* to be the location of creases on unfolded origami. Creases meet at interior vertices of the pattern; if  $n$  creases meet, we say that a vertex is of *degree*  $n$ . The angles between creases around a vertex in the pattern are called *sector angles*.

Each crease connects two facets, and creases are the edges of the *facet graph*. We say that any tree that spans the facet graph is a *facet tree*. Facet trees are easy to construct; any complete search method such as breadth-first or depth-first search is suitable. Creases not contained in a facet tree will be said to be virtually (but not necessarily physically) *cut* relative to that tree.

A facet tree implies a parent-child relationship between two facets connected by a crease. We will choose the convention that all facets are described by a counter-clockwise set of points in the pattern; we will associate a unit vector with each crease such that the vector's direction agrees with the order of vertices in the child facet. We then describe the *crease angle* as the angle between a parent facet and its child; the sign is chosen to be consistent with the 'right hand rule' applied to the crease vector.

Given a pattern and any facet tree, the crease angles associated with all uncut creases determine the kinematics of the origami mechanism – the pose of each facet and the angle of each cut crease can be determined by traversing the facet tree applying rotations to descendent facets.

Since origami is flexible and can be folded essentially flat, it is convenient to allow crease angles in the range  $[-\pi, \pi]$ . Flat origami illustrates a difficulty with a purely 'kinematic' model, however. Figure 3 shows the problem; although in the limit the crease angles of two flat origami pieces may be the same, the order in which facets are stacked is important. We will call a group of coplanar facets a *compound facet*. With each compound facet we associate a normal vector and a facet *stacking* relative to this vector. The height of a facet is its height in the stacking, and the height of a crease is the height of its child facet.

### III. NECESSARY CONDITIONS FOR FLAT-FOLDABILITY

We say origami is *flat* if all facets lie in a single plane. Origami patterns must satisfy some well-known necessary conditions in order to be flat-foldable; for example, *Kawasaki's theorem*, *Maekawa's theorem*, and *Meguro's theorem*. For discussion and proofs of these theorems, see

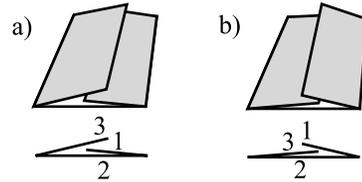


Fig. 3. Two nearly flat origami pieces.

Hull [5]. In this section, we present some extensions of these ideas based on the definition of the facet tree and stacking.

Flat origami has an important property: the rigid-body pose of each facet can be determined solely from the pattern, without knowledge of stacking or how to fold the pattern. We can *edge fold* any pattern by selecting any facet tree and traversing the tree, completely folding one uncut crease at each step and allowing facets to pass through one another where necessary. We say that an origami piece is 'edge-foldable' if, after applying edge folding, both sides of each cut crease line up. Flat origami is edge-foldable; this observation may be seen as a generalization of Kawasaki's theorem to multi-vertex patterns.

Maekawa's theorem provides a necessary condition on the crease angles around each vertex in the folded origami. If we know the stacking order for the goal, it is possible to do better:

*Theorem 1:* The facet graph and the goal stacking order completely determine which creases are mountain creases and which are valley creases.

*Proof:* Edge fold the pattern to find the facet normals in the goal state. Each crease connects two facets; the sign of the crease angle depends only on the normals of the two adjacent facets and the heights of the two facets. For example, if the normal to the parent facet points up, and the child facet is above the parent, the crease angle is positive. ■

There are further global necessary conditions for origami to be flat-foldable; the interested reader is referred to Hull [5].

### IV. ORIGAMI SKILLS

What skills do human origami experts use? <http://folds.net> lists some skills from simple to advanced. The first three categories are:

- Pureland: valley fold, mountain fold, turn over, rotate, book fold, ...
- Simple: reverse folds, prayer fold, ...
- Low-Intermediate: squash fold, petal fold, ...

Our version of the list omits some folds: pleat, radial pleat, cupboard fold, blintz fold, waterbomb base, preliminary fold, bird base, frog base, rabbit ear, and kite base. Each omitted fold can be folded by a sequence of the listed folds; for example, a blintz fold valley folds four corners of a square into the center. These compound

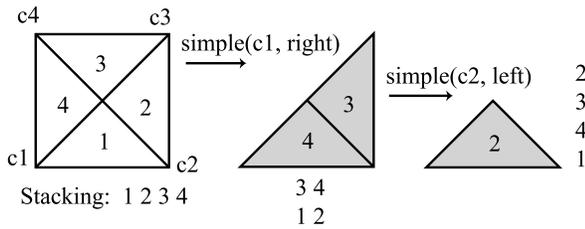


Fig. 4. A sequence of two simple folds.

fold do provide an advantage that we ignore in this paper – they allow *landmarking* (the center of a square can be precisely located by making a blintz fold), and also form larger building blocks for origami instructions.

The most basic skills are those in the ‘Pureland’ origami designed by John Smith, which allows only mountain and valley (‘land’) folds. (The motivations for the creation of Pureland origami were two-fold: the aesthetics of minimalist design, and making origami accessible to handicapped children; see <http://www.users.waitrose.com/~pureland/>.) With the exception of a few three-dimensional manipulations allowed in Pureland (primarily tucking and inserting of flaps), the fold models we present below are sufficient to describe Pureland origami.

#### A. Simple folds

The most basic origami fold takes all paper on one side of a crease line and folds it to the other side; the final pose of facets can be determined by reflection across the crease line. Figure 4 shows a sequence of two simple folds.

Given a flat origami state, what simple folds are possible? First, find the minimal set of lines that contains all creases. Discard any crease lines that cross a facet. Each remaining crease line divides the facets into two compound facets; we arbitrarily assign one to be the ‘base’ and one the ‘flap’. During folding, the base will not move. The flap can be folded either up (a *valley simple fold*) or down (a *mountain simple fold*).

To execute the fold, all creases colinear with the crease line are folded simultaneously. During folding, the heights of facets in the flap are reversed, and then either stacked above or below the base, forming a single new compound facet.

There are many interesting origami designs that can be simply folded. Figure 5 shows two of these: a cup, and a paper airplane.

#### B. Book folds

Figure 6 shows two examples of origami that cannot be simply folded. Origami folders often refer to the type of fold shown in figure 6b as a ‘book fold’ – the moving facet acts like the page of a book. We will define a book fold as follows: a *book fold* reflects a set of facets on one side of a set of colinear creases onto the other, such that

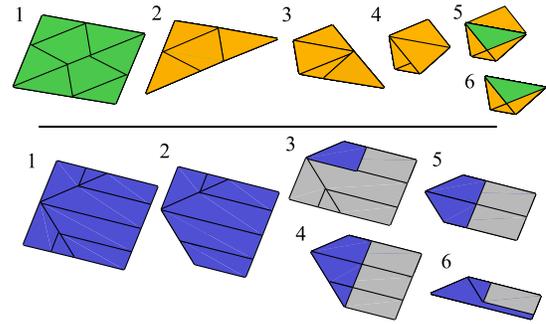


Fig. 5. Simple foldings of two traditional origami designs.

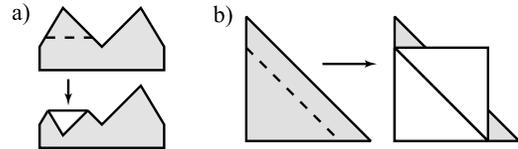


Fig. 6. Two designs that can be book folded but not simply folded.

the continuous rigid body rotation corresponding to the reflection does not cause self-intersection of the origami or tear any creases. We will call the moving facets the flap, and the fixed facets the base. We call the creases involved in the fold the *active creases*.

*Theorem 2:* The set of active creases in a book fold cuts the facet graph, separating the flap from the base.

*Proof:* Any crease that connects the base and flap and is colinear with the crease line must be active; the crease angle will be the angle between the base and flap, up to sign. Any crease that connects the base and flap but is not colinear with the crease line will be torn by any rigid body rotation of the flap around the crease line. ■

We can determine the set of all book folds from an origami state. First, enumerate all crease lines. Sort the creases by height. Consider all sequential combinations of creases that contain either the minimum- or maximum-height crease. Test each of these crease sets to determine if it cuts the facet graph into at least two pieces. All pieces strictly to the left of the current crease line are candidate flaps, as are all pieces strictly to the right. Test each candidate flap to see if and in which direction(s) folding is possible without self-intersection of the origami; this can be accomplished by polygon intersections in the plane of the compound facet.

Simple folding is a special case of book folding, and execution of a book fold is similar to execution of a simple fold – reflect the flap across the crease line, flip flap stacking, and stack the flap either above or below the base, depending on the direction of the fold.

## V. A PLANNER FOR BOOK-FOLDABLE ORIGAMI

Origami that can be folded using book folds is flat after each fold. Since the motion of the flap occurs out of the plane of the base, collision detection is only

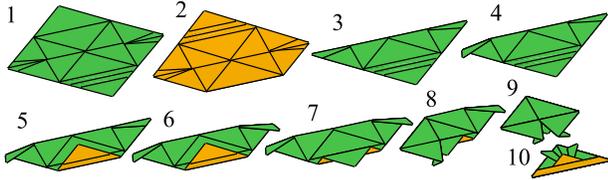


Fig. 7. Automatically planned folding of the samurai hat.

necessary at the beginnings and ends of folds, and only requires simple polygon intersection tests. Furthermore, the origami state after each fold is just the stacking of the facets, and is thus discrete. We have implemented a complete graph search planner for book-foldable origami; the nodes of the graph are flat origami states. Figure 7 shows an automatically planned folding of the samurai hat comprised of eight simple folds and one book fold.

The input to the planner is the pattern and the desired stacking of the facets. We now describe the algorithm. Use the goal stacking and the pattern to determine signs on the crease angles. Insert the pattern into the search queue as the initial state. While the search queue has elements, pop, test for goal state, and if goal, backchain to find the plan. Otherwise, determine the book folds from the state and generate successor states. Cull any states that have crease angles that do not agree with those of the goal state. Also cull any states that have been previously visited. Insert remaining states into the visited list and into the search queue.

The visited list is implemented as a hash table that hashes based on the integer heights of facets in the stacking. Before testing against the visited list, the compound facet of the state is collapsed to determine a *minimal stacking*. The algorithm to find the minimal stacking is essentially a bubble sort – each facet is allowed to bubble downwards in the stacking as long as it does not intersect with any facets in the level underneath it.

The planner implementation is about 5000 lines of C++ code, and was run on a 500 mhz Pentium III. The table below shows results for four traditional origami designs.

Origami	creases	nodes	time (sec)	folds
Cup	9	30	.1	5
Airplane	9	24	.1	5
Hat	14	75	.5	5
Samurai hat	20	4250	110	9

For the samurai hat, more than 99% of the CPU time was spent in polygon intersections to determine minimal stackings and to find book folds.

## VI. AN ORIGAMI FOLDING MACHINE

Figures 1 and 8 show a machine designed to allow a 4DOF Adept SCARA robot arm to make simple folds. A vacuum pad grips the paper, while the arm positions the paper over the folding device. The design is based



Fig. 8. A machine that can fold simple origami.

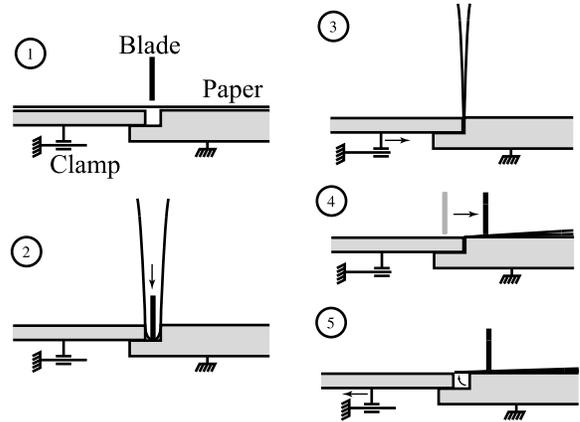


Fig. 9. Making a simple fold with the machine; side view.

on the following observation: any origami piece that can be folded by a sequence of flips and simple folds can be folded by a single initial flip and a sequence of valley simple folds.

The folding procedure is outlined in figure 9. The paper is first positioned on the table above a slot; the position of the paper determines the location of the crease. A blade presses the paper into the slot (step 2); friction holds the paper in the slot as the blade is removed. The slot clamps shut, forming the crease (step 3). Steps 4 and 5 show a method for removing the paper from the slot and placing it flat on the table; this is required since the arm used only provides one rotational degree of freedom at the wrist. First the blade sweeps across the paper, forcing it to lie flat. The clamp is released while the blade holds the paper against the table; the springiness of the paper allows it to swing free of the slot.

The primary goal of our design was to demonstrate that origami could be folded by machine. Error is inherent in the frictional insertion of the paper into the slot, and in the positioning and grasping of the paper, which are all open-loop. Typically, there is an error on the order of one millimeter in the positioning of each crease. Accumulation of errors limits the number of folds that

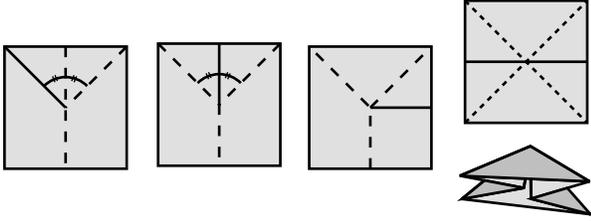


Fig. 10. Only the first design is book-foldable.

can be made reliably; nonetheless, the machine is able to completely fold both the simple airplane design shown in figure 5 and a simplified fourteen-crease version of the samurai hat – each design takes about two minutes to fold.

## VII. CLOSED CHAIN FOLDING

Many origami designs cannot be book folded; figure 10 shows some examples. Inside and outside reverse folds, squash folds, and petal folds all manipulate four creases simultaneously.

The planner for book folding described above relies on a key observation – self-intersection can occur only when the origami is flat. It turns out that a similar result holds for origami where four creases that intersect at a vertex are manipulated simultaneously:

*Theorem 3:* Rigid flat-foldable degree-four origami can only self-intersect when flat.

*Proof: (sketch)* First show that continuous motions of the origami cannot cause self-intersection without at least one crease angle reaching either zero or  $2\pi$ . To complete the proof, consider the cases under which a crease angle can become flat. Huffman [4] derives a relationship between opposite crease angles  $m$  and  $n$  for degree-four origami,

$$1 - \cos n = \frac{\sin A \sin B}{\sin C \sin D} (1 - \cos m), \quad (1)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are sector angles counter-clockwise from the crease corresponding to  $m$ . Kawasaki's theorem constrains  $A$ ,  $B$ ,  $C$ , and  $D$ ; for flat-foldable origami,  $n = \pm m$ . From this follows that if no two adjacent sector angles are equal or sum to  $\pi$ , and if one crease angle is zero or  $\pi$ , all of the others must be zero or  $\pi$  as well; the origami is flat. If two adjacent sector angles are equal or sum to  $\pi$ , book folding may be possible, but the theorem still holds. ■

Prayer folding and more advanced origami skills require the simultaneous manipulation of more than four creases. In the remainder of the section, we present the relationship between crease angles for vertices of arbitrary degree; our result is also applicable to the case where there is missing or excess sector angle around the vertex.

The mobility of a vertex of degree  $n$  is  $n - 3$ . We will therefore choose  $n - 3$  arbitrary independent crease

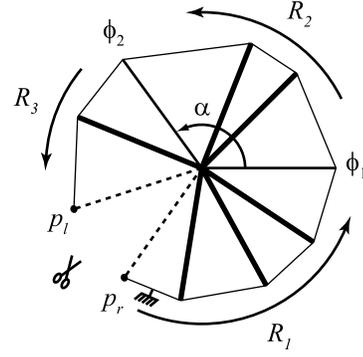


Fig. 11. Solving for three dependent crease angles.

angles as input, and solve for the remaining crease angles. (In the special case where the dependent crease angles are sequential, a simpler solution is possible using a reverse kinematics approach similar to that described in [3].)

Figure 11 shows the procedure;  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  are the crease angles to be solved for. First cut the crease corresponding to  $\phi_3$ . Anchor the next (counter-clockwise) facet, and choose a coordinate system so that the  $\phi_1$  crease falls on the  $x$  axis. The closure constraint can be written using a product of rotation matrices,

$$R_1 R_x(\phi_1) R_2 R_z(\alpha) R_x(\phi_2) R_z(-\alpha) R_3 p_l = p_r, \quad (2)$$

where  $R_i$  are rotation matrices corresponding to independent crease angles, and  $p_l$  and  $p_r$  are points that must touch to close the loop. Rewrite equation 2:

$$R_x(\phi_1) Z R_x(\phi_2) a = b, \quad (3)$$

where

$$Z = R_2 R_z(\alpha) \quad (4)$$

$$a = R_z(-\alpha) R_3 p_l \quad (5)$$

$$b = R_1^T p_r \quad (6)$$

are known. Multiplying out equation 3 gives three equations, the first of which is

$$k_3 = k_1 \cos \phi_2 + k_2 \sin \phi_2, \quad (7)$$

with known variables

$$k_1 = z_{12} a_2 + z_{13} a_3 \quad (8)$$

$$k_2 = z_{13} a_2 - z_{12} a_3 \quad (9)$$

$$k_3 = b_1 - z_{11} a_1. \quad (10)$$

If  $k_1 = k_2 = 0$ , then equation 7 implies that  $\phi_2$  can take on any value. Otherwise, equation 7 has the solution(s)

$$\phi_2 = \text{atan}(k_2, k_1) \pm \text{acos} \left( \frac{k_3}{\sqrt{k_1^2 + k_2^2}} \right). \quad (11)$$

There may be zero, one, two, or infinitely many solutions for  $\phi_2$ . For each value of  $\phi_2$ , the remaining two rows

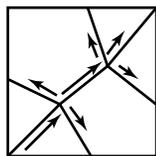


Fig. 12. A multi-vertex pattern with a mobility of one.

of equation 3 can be used to solve for  $\phi_1$ , which either has a unique value or is unconstrained.  $\phi_3$  is uniquely determined by the angle between the normals to the facets at either end of the cut chain.

### VIII. CONCLUSION AND ACKNOWLEDGMENTS

We have presented an articulated-rigid-body model that is rich enough to allow flat folding of origami. We have explored aspects of common skills used to fold origami, including mountain and valley folding, book folding (colinear creases), reverse folding, squash folding, petal folding (degree-four vertices), and prayer folding (high-degree vertices).

Future work includes an exploration of the ‘low-level’ manipulation skills (like landmarking) that humans use to precisely and reliably place creases in flexible paper. We also intend to explore further aspects of complex closed-chain manipulation. The configuration spaces of closed chains can be complicated, and understanding connectedness of these c-spaces is a very interesting problem; we have had some initial success in this direction using techniques based on those described in [13].

Another interesting problem is understanding multi-vertex patterns like that shown in figure 12. Huffman [4] points out that by viewing the vertices as nodes and the creases as edges, it is possible to find a relationship between crease angles for the network shown. The relationship between crease angles described in section VII allows this procedure to be extended to multi-vertex patterns with vertices of degree higher than four; in this case, there may be many ‘input crease angles’ in the network. Understanding the singularities and connectedness of c-spaces for these *compound* closed chains seems to be a challenging problem.

A final interesting result is that common paper shopping bags cannot be folded without bending the facets – our simple model of origami does not apply to what is probably the most common origami structure! Although we omit the proof, the open and folded configurations of the bag turn out to be isolated points in c-space. This could be considered a design feature; shopping bags tend to stay open since facets resist being bent. However, arbitrary out-of-plane flexing of paper facets is not possible, since paper does not stretch. (See work on developable surfaces and properties of paper; Kergosien *et al* [6] provides the most complete survey.) An interesting question is whether facets must crinkle as well as bend during folding of the shopping bag.

In current work with Demaine and Demaine, we have shown that a shopping bag *can* be flattened by adding a finite number of additional creases; this result gives an important boundary for the well-known *bellows theorem*, which states that closed flexible polyhedra have constant volume.

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### REFERENCES

- [1] E. D. Demaine and M. L. Demaine. Recent results in computational origami. In *Proceedings of the 3rd International Meeting of Origami Science, Math, and Education*, 2001.
- [2] S. Gupta, D. Bourne, K. K. Kim, and S. S. Krishnan. Automated process planning for robotic sheet metal bending operations. *Journal of Manufacturing Systems*, September 1998.
- [3] L. Han and N. M. Amato. A kinematics-based probabilistic roadmap method for closed chain systems. In *Workshop on the Algorithmic Foundations of Robotics*, pages 233–246, May 2000.
- [4] D. Huffman. Curvature and creases: A primer on paper. *IEEE Transactions on Computers*, C-25(10):1010–1019, Oct. 1976.
- [5] T. C. Hull. On the mathematics of flat origamis. In *Congressus Numerantium*, volume 100, pages 215–224, 1994.
- [6] Y. L. Kergosien, H. Gotoda, and T. L. Kunii. Bending and creasing virtual paper. In *IEEE Computer Graphics and Applications*, pages 40–48, Jan. 1994.
- [7] R. Lang. Trees and circles: an efficient algorithm for origami design. In *Proceedings of the 3rd International Meeting of Origami Science, Math, and Education*, 2001.
- [8] S. M. LaValle, J. H. Yakey, and L. E. Kavraki. A probabilistic roadmap approach for systems with closed kinematic chains. In *Proceedings of the IEEE International Conference on Robotics and Automation*, 1999.
- [9] W. Lenhart and S. Whitesides. Reconfiguring simple polygons. *Discrete and Computational Geometry*, 1994.
- [10] L. Lu and S. Akella. Folding cartons with fixtures: A motion planning approach. In *Proceedings of the IEEE International Conference on Robotics and Automation*, May 1999.
- [11] L. Lu and S. Akella. Folding cartons with fixtures: A motion planning approach. *IEEE Transactions on Robotics and Automation*, 16(4):346–356, Aug. 2000.
- [12] J. M. McCarthy. *Geometric Design of Linkages*, volume 11 of *Interdisciplinary Applied Mathematics (Systems and Control)*. Springer-Verlag, 1995.
- [13] R. J. Milgram and J. Trinkle. The geometry of configurations spaces for closed chains in two and three dimensions. *Homology, Homotopy, and Applications*, forthcoming; preprints available from author upon request.
- [14] S. Miyazaki, T. Yasuda, S. Yokoi, and J. Toriwaki. An interactive simulation system of origami based on virtual space manipulation. In *Proceedings of the IEEE International Workshop on Robot and Human Communication*, pages 210–215, 1992.
- [15] J. Trinkle and R. J. Milgram. Complete path planning for closed kinematic chains with spherical joints. *International Journal of Robotics Research*, 21(9):773 – 789, Dec. 2002.