Towards arranging and tightening knots and unknots with fixtures

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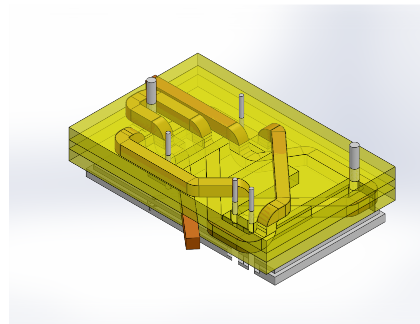
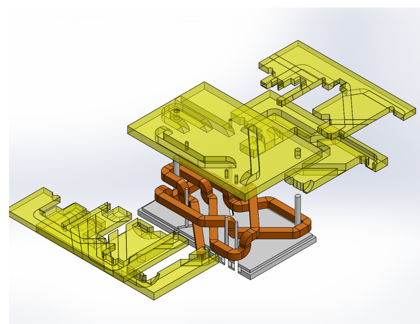
Abstract—**This paper presents a controlled tying approach for knots using fixtures and simple pulling motions applied to the ends of string. Each fixture is specific to a particular knot; the paper gives a design process that allows a suitable fixture to be designed for an input knot. Knot tying is separated into two phases. In the first phase, a fixture is used to loosely arrange the string around a set of rods, with the required topology of the given knot. In the second phase, the string is pulled taut around the tightening fixtures. Two sets of tightening fixture designs are presented. The first design is a fixture with no moving parts; tilted rods whose cross-sections get closer near the tips, guiding string in a controlled fashion as string slides up the rods during tightening. The second design is a collection of straight rods that can move passively along predefined paths during tightening. Successful tying is shown for three interesting cases: a "cloverleaf knot" design, a "double coin" knot design, and the top of a shoelace knot.**

Note to practitioners: This paper shows how simple mechanisms, requiring only very limited actuation, and no sensing, can be designed that allow knots to be tied automatically in string or rope in two steps. In the first phase of knot tying, the string is loosely arranged into the shape of the knot using pressurized air. In the second phase, the ends of the string are pulled to tighten the knot in a controlled fashion, guided by the geometric structure of the mechanism. Although these devices are intended as a proof of concept, the automatic design process can design these mechanisms for a broad family of knots, and experimental work shows that simple knots can be arranged and tightened in a few seconds.



(a) Fully assembled shoelace tying fixture.



(b) The fixture being disassembled, exposing the string and rods. The orange tube shows the shape of the string.

Fig. 1: Models of assembled and disassembled arrangement and tightening fixtures for the shoelace unknot.

I. Introduction

Knots are used for practical binding tasks (shoelaces, climbing knots, surgical suturing) and decoration (ties and bow ties, gift wraps, pendants). This paper shows an approach to tying a variety of knots automatically using simple control (pulling the open ends of string) without sensing. This paper expands on a paper originally presented at the 2014 Workshop on the Algorithmic Foundations of Robotics [47]. We do not focus on a particular application of knot tying, but rather on the underlying manipulation problem.

In the presented approach, a fixture first constrains the string so that simple actuation can be used to loosely *arrange* the string into a shape with the desired topology around a set of parallel rods. This outer *arrangement fixture* is then disassembled to expose the string and the rods. These parallel rods form part of a *tightening fixture* that passively leads the string (in a controlled fashion) to a desired tightened configuration when the ends of the string are pulled. Figure 1 shows how the six pieces of the arrangement fixture (transparent yellow) may be disassembled to expose the string and the tightening fixture (gray rods). Figure 3 shows a physical implementation of a knot-tying fixture.

Simply pulling the open endpoints of string after arrangement of a knot can lead to incorrect tightening of complex knots for which friction locks are intended to occur at particular locations along the string; Figure 2b) shows an example of what can happen to a *cloverleaf knot* (Figure 2a) during simple pulling. Humans appear to ensure correct tightening using a complex sequence of maneuvers and placement of extra fingers. Our approach is motivated by the idea that a fixture can take the place of extra fingers and control, allowing correct tightening simply through pulling on the ends of the string. In Section IV-D, a cloverleaf knot is shown to be tied using the fixture designed.

We have explored two methods of tightening. The first tightening fixture design (Figure 11) is composed of stationary sloped rods that guide the string during tightening. The second fixture design (shown in Figure 17) allows the rods to slide closer together during a controlled tightening process. The tying of different knots is shown in the video attachment, which will demonstrate the general process of our knot tying process with the assistance of the fixtures.

The basic idea of the arrangement fixture (without any capability for tightening) was presented first in the thesis of coauthor Bell [11], and is explored experimentally in [12]. The primary contribution of the current paper is the significant extension of this work to show a complete system for tightening as well as arrangement, and proof-of-concept application to knots that are more complicated than those previously explored.

The fixture-based approach admits reliable knot tying, and the paper will make some arguments as to why the approach can be applied to a broad family of knots. In fact, the approach can even be applied to "unknots" such as the top of a shoelace knot, for which pulling on the ends of un-fixtured string would actually untie the (un)knot. The paper will present a process that allows a fixture to be designed for a given knot, and will show examples of fixtures for the double-coin decorative knot, a shoelace unknot, and a cloverleaf knot.

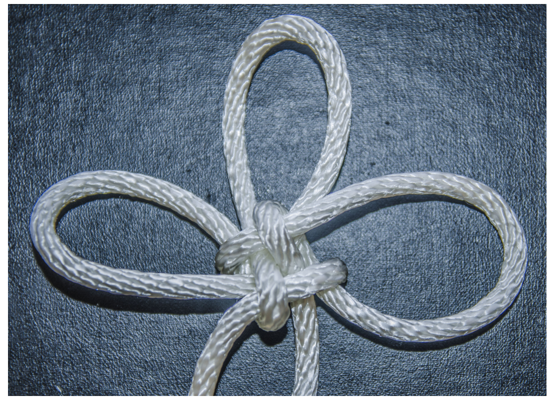
In manipulation, we sometimes have the luxury of designing a mechanical process so that simple models are sufficient to describe the behavior we care about. This principle drives the fixture design. We avoid modeling the string as a general (and unpredictable) continuous 3D curve by first ensuring that the configuration of the string achieves the desired topology using the arrangement fixture. Then the string is pulled tight around the tightening fixture, and takes on an essentially polygonal shape; this polygon may be computed by considering the shortest curve for the string in a homotopy class enforced by the rods.

Undesired friction between strings might cause premature *friction lock*—high friction at certain contacts that prevents further tightening of the knot. Our strategy to avoid the necessity of modeling unpredictable string-string friction contacts is to tighten knots in a controlled fashion using the fixture, so that string-string contacts are delayed until they become necessary (and desired) in the tied configuration.

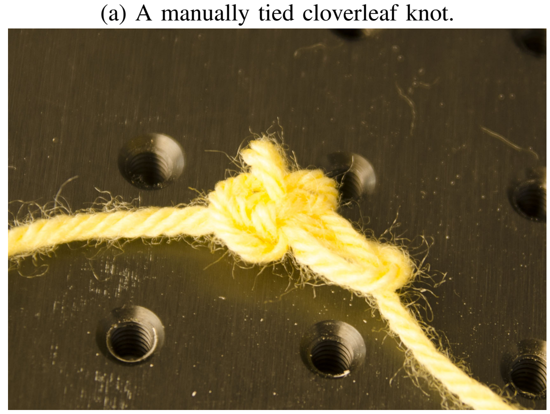
Although a new fixture needs to be designed for every new knot, we describe an automated design process in Section V which outputs the layout of the fixture. A human then uses basic 3D extrude and cut operations in a 3D modeling package to transform the layout to a physical model suitable for printing. In our experience, this process takes about 10 minutes of work by a human. Fixtures of the size discussed in this paper are typically printed in a few hours on the Objet Eden 250 printer we used.

Although we expect that the approach would be effective for string of various thicknesses, current work has focused on tying yarn of only two thicknesses.

Even thought the proposed approach is the first we are aware of that can tie (and eventually tighten) different kinds of knots, it has various limitations. Due to the size of the fixture, the proposed method is not capable in constrained spaces. In Section III-B, we will show that our fixture is able to arrange and tie knots around objects, but only around simple objects which contains parts that can be bounded by a cylinder, such as rods or rings. In addition, the knot tying process is not as fast as we would like to see in a factory. Even though this approach can be used to tie many common knots, even the ones that consists of multiple strands of string, the approach depends on the delay of the string-string contact. For many knots where the contacts is part of knot formation, such as the knots shown in Figure 13, they cannot be tied using this approach.



(a) A manually tied cloverleaf knot.

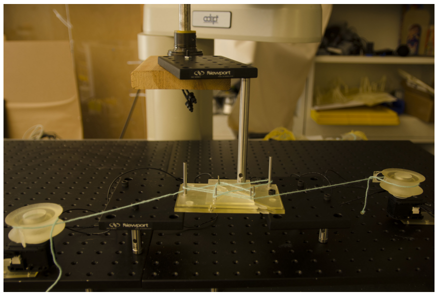


(b) A cloverleaf knot tightened (incorrectly) by pulling the ends of the string without fixturing.

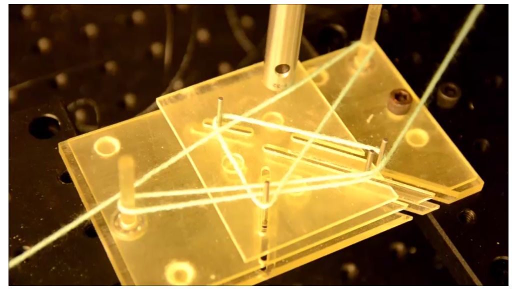
Fig. 2: Two cloverleaf knots tied by hand.

Section I-A will show the overall approach using the example of a shoelace unknot using the first tightening fixture design. Section II briefly examines mathematical models of knots and unknots; these models are the basis for discussion of both arrangement and tightening fixture designs.

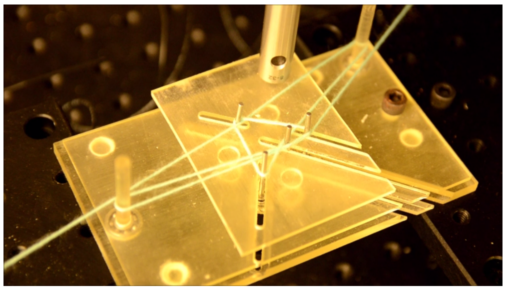
Section III discusses separable arrangement fixtures. Although the knots studied in this paper are more complex, the arrangement process is essentially the same as that presented in our previous work on knot arrangement [12]. For this reason, our description of knot arrangement with fixtures in Section III is brief. Rather, we focus on showing how arrangement fixtures may be designed to embed objects that the knot is intended to tie around, such as tightening fixture as well as the string. Section IV describes the design and improvements of tightening fixtures. The complete fixture system and the unified design process are presented in Section V using a *double coin knot* as an example.



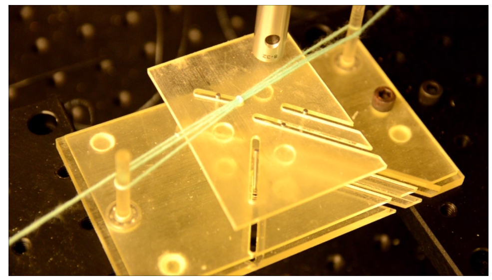
(a) The tightening fixture system along with motors and arm grippers.



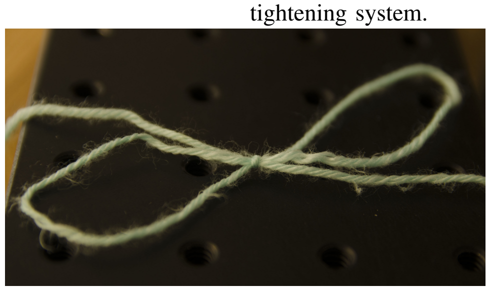
(b) The starting configuration of the knot around the tightening system.



(c) The knot is pulled tight around the rods, and the sliding rods cannot move any further.



(d) The final configuration of the knot tightened using the passive tightening system.



(e) A shoelace unknot tightened by the fixture.

Fig. 4: Tying the shoelace (un)knot. For clarity, in Figure 4b, the rods are colored blue.

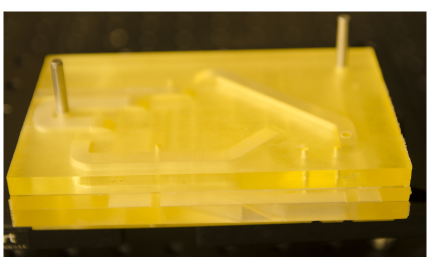


Fig. 3: Assembled knot-tying fixture for shoelace (un)knot.

A. Example: tightening the shoelace unknot

Figure 4d and 4e show the result of arranging and tightening the top of a shoelace knot (which we will call a *shoelace unknot*) using a fixture designed for the purpose using the general design principles that will be presented later in the paper.

The tightening is accomplished in four stages. First, the fixture is assembled, as shown in Figure 1a; pressurized air pushes the string through the fixture to obtain the desired topology. The configuration of the string around the rods is bounded by the (orange) tube shown in Figure 1b. The second step is to separate the arrangement fixture (yellow transparent pieces in Figure 1a) to expose the string around the rods in a loose configuration, which is shown in Figure 1b. In Figure 1b, the thin straight gray rods can move along the designed track, while the thick straight gray rods remain stationary.

The third step is to pull the string as the string slides along the rods, and the thin rods slides towards the center of the tightening fixture, until the knot is nearly tight (Figure 4c). The last step is to lift the string up from the thin rods, and pull to finish tightening (Figure 4d).

The basic geometry of the fixture was found using the design process which will be described in section V; further work by a human engineer using Solidworks thickened the various tubes and rods to create the complete 3D model of the fixture. The fixture shown in Figure 3 was then printed using an OBJET Eden 3D prototyping machine.

We used two Dynamixel MX-28 servo motors to pull the ends of the string. Figure 4a shows the placement of motors. The last step of lifting the string off the think rods was achieved by using an Adept Cobra Robot Arm, though a simpler system that provides one translation degree of freedom could replace the arm.

When the knot is lifted from the thin rods, the "center" part of the shoelace unknot is free while the two "ears" are still spanned by rods. When pulled, the "center" of the shoelace can then be fully tightened. We repeated the procedure ten times, and all trials successfully tightened a shoelace unknot.

B. Related work

To our knowledge, this is the first work on unified principles that allow the design of fixtures for tying and tightening a broad family of knots. However, the work builds strongly on work on knot-tying from a broad set of background areas. This section provides only a brief survey; studies related to string manipulation and knot tying are discussed in more detail in [11].

Even though we only study how to tie knots with physical string in this work, other material can be folded (or tied) into knot like structures, such as folding proteins [21].

A brief introduction to mathematical knot theory may be found in [1], [2], [34], [43]. In particular, one central aspect of mathematical knot theory is the study of *knot invariants*: properties that hold across a set of geometric curves that we might consider to all represent the same knot. This study is particularly relevant to the design of knot fixtures: controlling the geometric configuration of the string exactly is usually hard, but achieving the correct knot topology may not be so difficult.

Work in the knot theory community also studies how to untie unknots [27], [28], [29], [30] such as the top of a shoelace knot. (Unknots will be discussed in more detail below.) If we would like to achieve a particular geometric configuration of an unknot, we must specifically prevent untying motions during reconfiguration of the string.

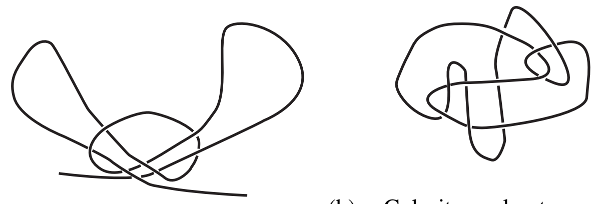
The study of tightness of a knot is also a very interesting area of applied mathematics and physics [40], [10], [36]. Recently, a *rope length* parameter has been used to study the tightness of a knot in rope of a given thickness [6].

Engineers have designed many different machines to tie [18], [19], [44] and tighten [45] different knots, but designs are typically complex, and are specific to particular knots.

Recently, elastic rods have been used as a model for manipulation of wire [17]. Flexible needles have been applied to manipulate string [4], [3]. String manipulation has also been explored as a challenge problem for single- and multi-arm coordination [32], [33] .

The configuration of string wrapped around rods in our tightening fixtures may be modeled as the shortest curve among point obstacles; algorithms to find such curves have been studied in the computational geometry community [14], [20], [25], [26], [31]. In previous work, we optimized the layout of the string in arrangement fixtures using graph drawing algorithms [46], [23]. To find the tension along the string after wrapping around frictional rods, the capstan equation [7], [24] may be of some use.

Although we are only beginning to understand approaches to tying knots with fixtures, our approach to this problem draws inspiration from problems previously studied in the context of manipulation and motion planning. Both the arrangement fixture and the tightening fixture are used to cage [16], [41] the string (although in different ways), since exact control may not be possible. One could study the frictional contact modes [9], [37], [13] between the string and the rods, or between string and string. Or perhaps avoiding friction altogether would be wiser; adding low-friction ball-bearings to areas inside the arrangement fixture and at locations along the tightening fixtures can simplify tightening, using ideas like those explored by Furst and Goldberg [22]. Restricting paths to certain homotopy classes becomes critical in the fixture design, since precise control may not be possible; recent work by Bhattacharya *et. al.* [15] explores algorithms for generating such paths in the context of motion planning. The focus on tight string recalls work on using minimum energy configurations of flexible bodies for the purpose of motion planning [38], [42].



(a) Shoelace unknot. (b) Culprit unknot, redrawn from [30].

Fig. 5: Knot diagrams of unknots.

Early work in manipulating cartons by Lu and Akella [35] served as one inspiration for current work, as did work on robotic folding of origami [48], [8]. Fixtures have also recently been used to manipulate ribbons used in cancer treatment [39].

II. Mathematical knots and knot diagrams

What is a knot? What types of knots can be arranged or tightened by fixtures? Our general approach to designing fixtures is based on the idea of "knot diagrams", representations of knots developed in the mathematical knot theory community. Conveniently, knot diagrams are planar, and show knots in loose, rather than tight, configurations. We therefore use knot diagrams with the desired geometry as a starting point to design arrangement fixtures, and to place the rods in the tightening fixtures. This section describes mathematical models of knots, and the relationship between knot diagrams and our fixtures.

A mathematical knot is an embedding of a circle in $\mathbb{R}^3$ with no open ends. A mathematical knot cannot be untied (the topology of the string cannot be changed) without cutting the circle. Figure 6 shows a trefoil mathematical knot, which is an overhand knot for which the open ends have been glued together.

If we glue together the open ends of the shoelace tied in the previous section, we get what is referred to as an *unknot*. Technically, an unknot is a circle or any of its ambient isotopes in $\mathbb{R}^3$; informally, some moves can be applied to the unknot

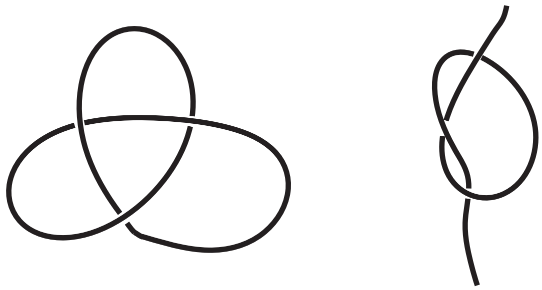


Fig. 6: A trefoil knot (on the left) is a mathematical knot. When the edge is cut, the knot becomes an overhand knot (on the right) that we can physically tie.

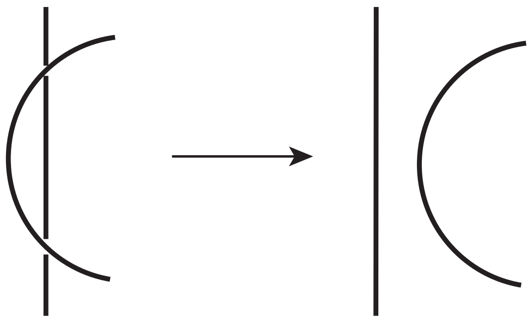


Fig. 7: An example of a type II Reidemeister move.

that "untie" the unknot without causing self-intersection of the string and without cutting the string.

A knot diagram is the regular projection of a knot (or unknot) to a plane with broken lines indicating where one part of the knot under-crosses the other part. We refer the points indicated by the broken lines as *crossings*. Figure 5 shows knot diagrams for the simple shoelace unknot, and for a more complicated unknot, the *culprit unknot* studied in [30].

Although knot diagrams for a particular knot are not unique, Reidemeister moves [5], corresponding to specific manipulations of string near the crossings, can be used to transform between any pair of knot diagrams for a given knot type. For an unknot, there always exists a sequence of Reidemeister moves that transforms the knot diagram into a simple loop. For example, Figure 7 shows a type II Reidemeister move, which removes two consecutive over-crossings.

In contrast to mathematical knots, physical knots have open ends, but by immobilizing the ends of string (in this paper, using the motors that tighten the string), a given topology consistent with that of a mathematical knot can be maintained. Arrangement fixtures, by arranging the string into a geometric configuration with a desired sequence of crossings given by a knot diagram, therefore provide a good starting point for forming a tightened knot.

The situation with unknots is more complicated. When you pull to untie your shoelaces (Figure 5a), two Reidemeister moves are in some sense physically performed to remove some crossings. However, if you hold the two loops ("ears") of the shoelace knot in place with your fingers (thus preventing

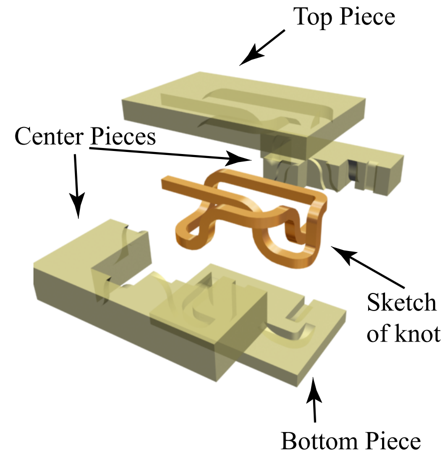


Fig. 8: An example of a four-piece arrangement fixture for an overhand knot, from [11].

Reidemeister moves from happening) while pulling the open ends, you will prevent the shoelace from being untied, and in fact, perhaps further tighten the central part of the knot. The embedded rods serve the purpose of preventing Reidemeister moves for unknots; the rods allow a particular desired topology of the string (around the rods) to be guaranteed even for unknots.

III. Knot arrangement with fixtures

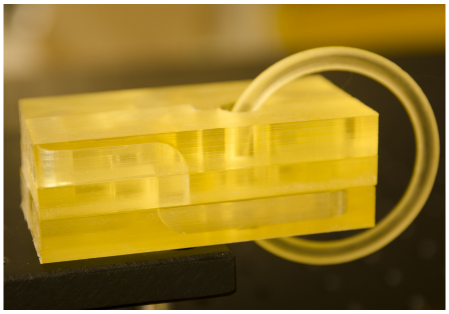
This section describes the approach to designing fixtures that arrange string into the desired knot topology.

A. Arranging knots: separable four-piece fixtures

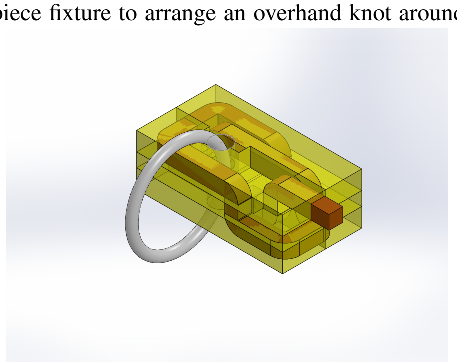
Bell's thesis [11] proved that *any* knot can be arranged in such a way that a fixture can be designed so that fixture may be disassembled in a very simple way to extract the knot:

Theorem 1. *[11] Given any (mathematical) knot consisting of one or more strands of string, and described by a Gauss code, a fixture can be constructed that loosely arranges string into a (physical) knot with the same Gauss code, provided that the endpoints of the string are connected together outside of the fixture. Furthermore, this fixture can be cut into four pieces in such a way that all four pieces can be removed by pure translation without colliding with the string.*

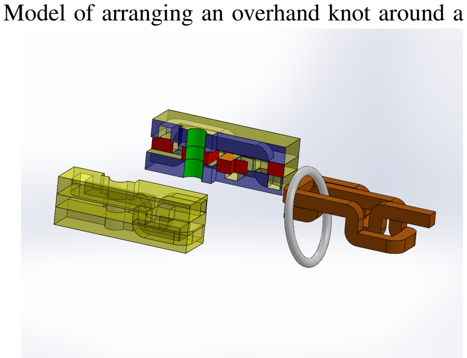
Figure 8 shows an example of a constructed fixture. Although we omit the proof of the theorem, we summarize the approach, since we will need it to describe how to build easily disassemble-able arrangement fixtures around rods. We assume that the string has some non-zero thickness. The Gauss code is the sequence of junctions, each of which is denoted as an over-crossing (o) or an under-crossing (u), as we walk along the string [12].. First, make a cut in the knot diagram between each pair of crossings, dividing the curve into sections that each contain a single crossing, either "over" or "under". Call the set of sections with over crossings the top layer, and call the collection of sections with under crossings the bottom layer. Lift the top layer vertically out of the page, in the z



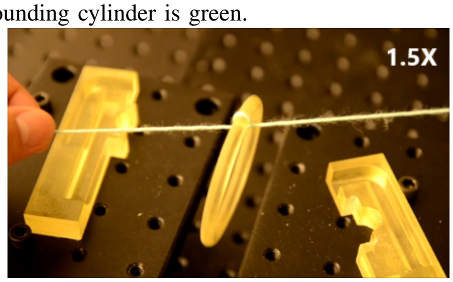
(a) Six piece fixture to arrange an overhand knot around a ring.



(b) Model of arranging an overhand knot around a ring.



(c) Disassembled arrangement fixture, exposing the string (orange tube) around the ring. The cut surfaces are colored blue and red for different layers, while the surrounding cylinder is green.



(d) Overhand knot tied around a ring by fixture.

Fig. 9: Arranging an overhand knot around a ring.

direction. Now connect the two layers using additional vertical sections of string; call the layer containing these vertical sections the middle layer.

Building an easily-separable fixture around this new curve is now straightforward. Build a top piece that covers the top layer, a bottom piece under the bottom layer, and surround the vertical sections by fixture material as well. The top and bottom pieces may be removed from the string using pure translations (up and down respectively), and the middle layer can be cut into two pieces and removed using sideways translations.

Even though we can always separate the top piece and the bottom piece, the middle layer cannot always be immediately cut into two pieces and separated using sideways translations. Some rearrangement of the vertical sections of curve may necessary to ensure that only two pieces are needed to extract the middle layer of string.

For example, imagine that we look at the arrangement of string from the side. If all vertical sections of string in the middle layer are visible from the view direction, then two separable pieces of fixture can be used to cover the vertical sections: one covering the visible front sides of the string, and one covering the back sides. If some sections are not visible (obstructed by other vertical sections), then two pieces of fixture may not be sufficient. Therefore, we sometimes need to rearrange these vertical sections to ensure the separability of the middle layer.

B. *Arranging knots around objects*

In this section, we show how to extend the four piece fixture design to support tying knots around different objects. Topologically, perhaps the most interesting object to tie string around is a ring, since the top and bottom pieces of the fixture cannot be extracted from the ring without cutting the fixture.

For simplicity, we find a vertical bounding cylinder (colored green in Figure 9c) around the ring (or other object), and embed the bounding cylinder into the fixture. The middle layer of the four-piece fixture is cut into two pieces to allow the fixture to be separated from vertical sections of string without interference. If vertical cylinders are placed along this *cut surface* (colored as blue or red in Figure 9c for different layers), the middle section of the fixture will be separable without interfering with the cylinders. However, the cylinders are longer than the vertical sections of string, and extend into the top and bottom layers of the fixture. To allow separation of these sections, the cut in the middle layer may be extended into the top and bottom layers; the fixture now has six pieces.

Figure 9d shows a successful tightening of an overhand knot around a ring, using our six-piece fixture design.

C. *Russian dolls: embedding a tightening fixture in an arrangement fixture*

In the first phase, the arrangement fixture guides the string into a loose knotted configuration. In the second phase, the tightening fixture guides tightening of the loosely knotted string. How can we transfer string between the two fixtures?

Just as tiny Russian dolls can be stored inside of larger hollow Russian dolls, we can embed the tightening fixture inside of the arrangement fixture with careful design. Then, during arrangement, the string is arranged around the embedded tightening fixture; the arrangement fixture can then be removed, leaving the string in position for tightening.

A side view of a six piece fixture used to arrange string around a tightening fixture is shown in Figure 10. The separation sequence is similar to that of the four-piece fixture, except that the top and bottom sections require some sideways translation (to be removed from the rods) after the initial upwards and downwards translations (which are used to separate from the string).

If there is a direction from which all rods and vertical string are visible, then a cut surface can be computed to allow separation from both rods and vertical string. The direction from which all vertical elements (rods and vertical string) can be seen allows translation of a "front" piece of the fixture encasing the vertical elements towards the viewer, and a "back" piece can be translated directly away from the viewer; the union of these front and back pieces is the complement of the vertical elements.

Let there be some desired *visibility direction* in the *x*-*y* plane; we can detect if rods or string are visible by projecting their locations onto a line orthogonal to this visibility direction, as shown in Figure 20b. Based on this approach, we can successfully design and embed a tightening fixture into the arrangement fixture, such as the one for shoelace unknot shown in Figure 11.

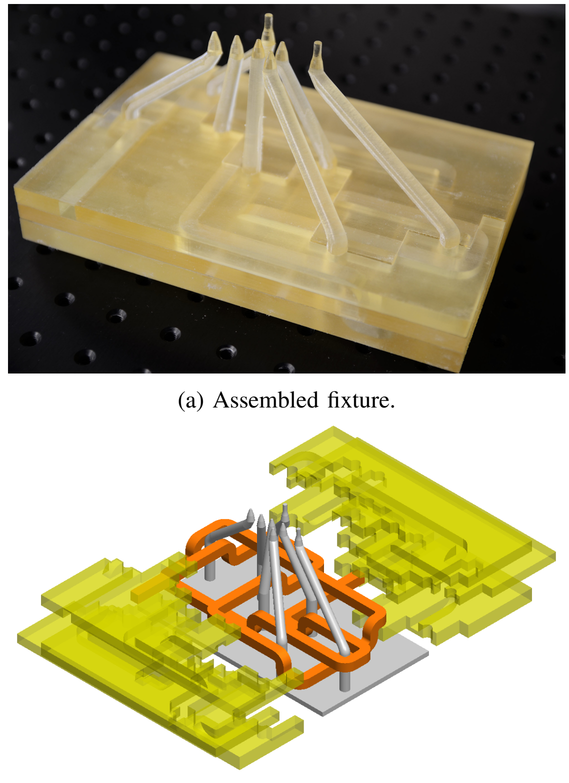
In Figure 11, conical tips on some of the tightening rods are designed to let the string escape the fixture and reach a final tight configuration, since the rods cannot be arbitrarily thin or be arbitrarily close. The video attachment shows the usage of this tightening fixture, with the string escaping the conical tips.

What if there is no direction from which all vertical elements (rods and vertical string sections) are visible? Since we are only concerned with attaining a desired topology of the string around rods, we may slide the vertical elements in directions parallel to the projection line until visibility from the projection line is achieved, as shown in Figure 20c.

IV. Controlled tightening using fixtures

We usually consider a knot "physically tied" only when the knot has been tightened to the point that application of certain (perhaps bounded magnitude or constrained direction) forces does not further change the location of any string-string contacts, as measured along the string. Typically, we expect some friction lock to appear at the end of the tightening process; such friction lock prevents further motion of the contacts. The main idea we make use of is to build tightening fixtures that delay string-string contact during tightening until the knot is "tight enough" to achieve the desired friction lock by simply pulling on the endpoints.

We present two approaches to designing tightening fixtures. The first approach has no moving parts, and the string simply slides along some rods, each with a particular slope designed



(a) Assembled fixture.

(b) Disassembled fixture.

Fig. 11: Assembled and disassembled arrangement and tightening fixtures for the shoelace unknot. The orange tube in subfigure (b) shows the shape of the string after arrangement but before tightening; the gray rods are the tightening fixture, and the yellow transparent parts are separated arrangement fixture.

to guide the knot to tighter configurations. This approach has the strength of simplicity, and such fixtures can be printed using a rapid prototyper with no additional work. However, friction between string and fixture can become a problem for more complex knots. The second approach uses rods that slide closer together during tightening, and spin to reduce the effect of friction. This more-complicated mechanical design does pay off in faster, more reliable tightening.

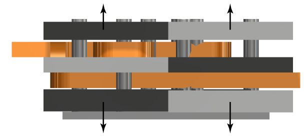
Both styles of tightening fixtures include a set of carefully arranged rods. Let us define a **cell** of a knot diagram as an open bounded connected region of $\mathbb{R}^2$ enclosed by the knot diagram; it is a subset of the complement of the knot diagram. For simplicity, we typically place one rod in every cell; this is sufficient to prevent Reidemeister moves and thus preserve the intended crossings, even for unknots.

A. Single-piece tightening fixtures

We first discuss tightening fixtures with no moving parts.

The straight sections of the rods embedded in the arrangement fixture ensure desired crossings. The tilted sections of rods allows controlled tightening, for the subset of knots for which the desired final configuration is close to planar. We give no formal definition of "close to planar", but as an example,

(a) Side view of arrangement fixture assembled around tightening fixture.

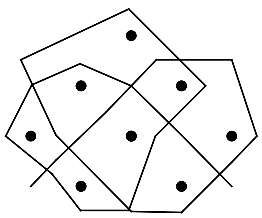


(b) Disassembly of top and bottom of arrangement fixture by vertical translation.

(c) Disassembled arrangement fixture and exposed string.

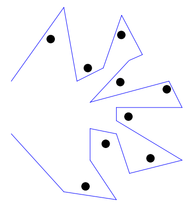
Fig. 10: Disassembly sequence for arrangement fixture.

(a) Input to the design process, the desired geometry of a double coin knot.

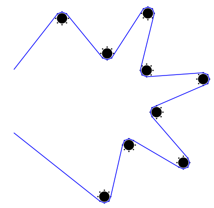


(b) Rod placement in cells.

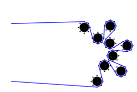
Fig. 12: Double coin knot.



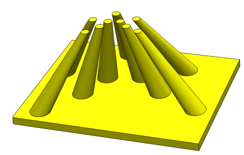
(a) Loose string around disc obstacles.



(b) Tight around disc obstacles.

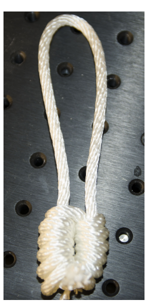


(c) Reconfiguration of discs.

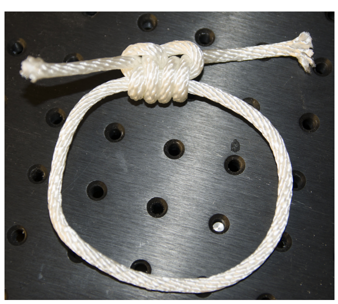


(d) Fixture to tie star shape.

Fig. 14: String tightening around set of rods.



(a) A padlock knot.



(b) A door knocker knot.

Fig. 13: Non-planar knots, such as those shown here, cannot be tightened using our fixtures.

Figure 12a shows a decorative *double coin knot* that is nearly planar.

Figure 13 shows a few non-planar knots that cannot be tied using our fixtures. These knots contains wrapping around a segment of string, which is difficult to achieve using a planar fixture.

How should these tilted rods be designed? Let the final configuration of the knot lie (approximately) in an *x*-*y* plane. Imagine also that the initial, looser configuration of the string lies in the same *x*-*y* plane. Now let us consider how the string would need to move for tightening within this single plane.

Between the initial and the final time, choose a continuous (w.r.t. time) family of configurations of the string in the x-y plane. For our purposes, we choose the initial configuration to be a linearly-scaled version of the final configuration, and choose intermediate configurations based on linear interpolation. Place small disc-shaped obstacles around the string so as to constrain the shape of the string to a polygon with approximately the correct shape (we will discuss this detail in subsection IV-B below). Figure 14b shows such an initial configuration, and Figure 14c shows a final configuration, both constrained by disc obstacles. If we move the discs between initial and final configuration while tightening the string, the string will be constrained to move to the final configuration as well.

The motion of the string can now be described as a surface in (x, y, t) space-time; the $t = 0$ and $t = 1$ slices of this surface correspond to initial and final configurations of the string. Based on our previous assumption, each later slice can be achieved by moving obstacles inwards (using the same linear transformation applied to the string) while pulling on the ends of the string to tighten, assuming no friction between string and obstacles.

In space-time, the motion of the disc obstacles generates a set of tilted rods ("space-time obstacles"), and we can build a physical model of these rods, by substituting the z dimension for the time dimension. We can use the rods to tighten knots by wrapping the string around the base of the rods (physically $z = 0$, corresponding to $t = 0$ in space-time), and pulling on the ends while moving the rods in the z direction. Figure 14d shows an example of such fixture to tighten the star shape.

B. Shortest curves around point obstacles

How should the rods be placed to guarantee the intended polygonal shape of the string, with sufficient clearance between rods and string so that the arrangement fixture can be assembled around both?

A simplified version of finding how string wraps around obstacles in the plane has been studied in computational geometry—finding the shortest curve within a homotopy class described by a set of *point obstacles* [14], [20], [25], [26], [31], given the coordinates of the point obstacles and the polygonal curve that describes the initial "loose" configuration of the curve in certain homotopy class. This section uses shortest curves in a homotopy class to model the shape of the string as the string is pulled around the tightening fixture.

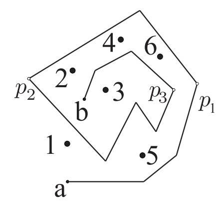
Cross sections of rods are discs rather than point obstacles, and the radii of the rods matters as the string approaches its tightest configuration. Therefore, we used a set of point obstacles on a circle to approximate the shape of the disc cross-section to compute how we expect string to wrap around the tightening fixture, and to inform the design of the tightening fixture.

To compute shortest paths in a homotopy class around points, we used the algorithm presented in [14]; because this algorithm was unfamiliar to us, and may be unfamiliar to others in the robotics community, we outline it briefly below.

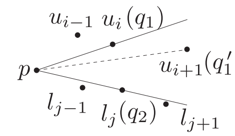
The input to the algorithm is a (presumably loose) polygonal path around the point obstacles. First, divide the polygonal curve into a set of x monotone paths where each path has either non-decreasing x coordinates or non-increasing x coordinates, such as segments $ap_1, p_1p_2, p_2p_3, p_3b$ in Figure 15a.

The first significant task of the algorithm is to find a *canonical representation* of each *x monotone path*, describing the relationship of each path to the point obstacles. We describe each x-monotone path using its relative position to all point obstacles. For example, the canonical representation of path ap_1 and p_2p_3 are $1^+,2^+,3^+,4^+,5^+,6^+$ and $1^-,2^+,3^+,4^+,5^-,6^+$ respectively, assuming each point obstacle is labeled with $k \in \{1,2,\ldots,n\}$, with a $^+$ sign if it is above (otherwise $^-$) the corresponding monotone path.

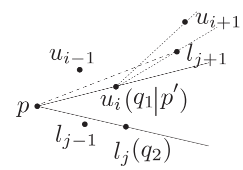
Using the canonical representation, we can represent the entire polygonal path by patching up the representation for each x-monotone path in sequence. We can then simplify such representation by deleting the relative information about a point obstacle if the same point obstacle appears with same sign twice consecutively in the representation. For example, between p_1p_2 and p_2p_3, obstacle 1 is under both x-monotone paths, labeled as 1^- at the end of p_1p_2 and at the beginning of p_2p_3. Therefore, we can simplify the description by deleting the two adjacent 1^- symbols in the representation. The canonical representation of the curve described in Figure 15a is $1^+,2^+,3^+,4^+,5^+,6^+,6^-,5^-,4^-,3^-,2^-,2^+,3^+,3^-$ after simplification.



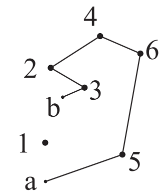
(a) Polygonal input of a curve, obstacles and x-monotone paths.



(b) The case of p not updated.



(c) Case of p gets updated.



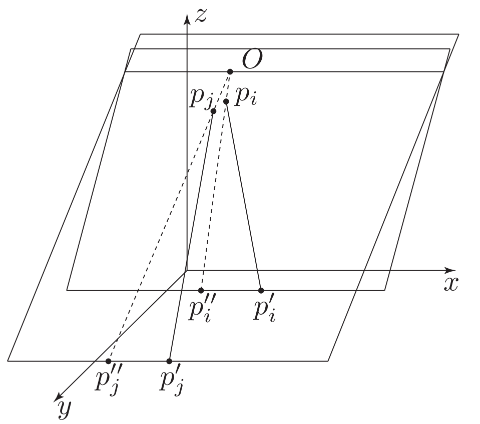
(d) Shortest homotopy curve of example in Figure 15a.

Fig. 15: The illustration of the search for shortest homotopy curve.

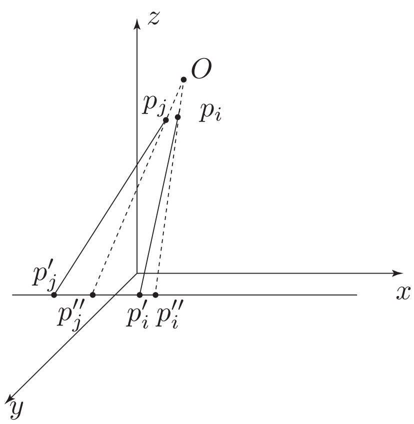
For each monotone path π starting at p, we want to find the shortest admissible curve from p to the end of π between U and L (let B_U denote the points (obstacles) above π, and B_L the points under π; let U be the lower envelope of B_U and L be the upper envelope of B_L) such that the shortest curve is within the same homotopy class as π. By lower (upper) envelope of B_U (B_L), we mean a subset of points in B_U (B_L) with different x coordinates such that by connecting every two points in the subset with adjacent x coordinates with a straight line segment, the rest of points of B_U (B_L) are all above (under) the connected polygonal path.

The basic idea is based on visibility, and we refer the area with apex at p and between U and L as a funnel. Let u_i and l_j be the ith and jth points on B_U and B_L. If there exists a straight line between p and u_{i+1} (or l_{j+1}) in the funnel, then the shortest curve up to u_{i+1} (l_{j+1}) will be a straight line, as in Figure 15b.

If pl_{j+1} (or pu_{i+1}) intersects with U or L, consider the funnel apex at p shown in Figure 15c. Find $q_1 \in U$ and $q_2 \in L$ such that all $u_k, k \leq i$ are above or on pq_1 and all $l_k, k \leq j$ are below or on pq_2, as the example shown in Figure 15c. If pl_{j+1} is above pq_2 (as shown in Figure 15c), then q_1 (q_2 if pu_{i+1} is below pq_1) becomes the new apex p, and the shortest curve up to q_1 (q_2) is recorded. Figure 15c shows the case when q_1 becomes the new apex. Repeat until the end of π, and apply the previous procedure to all monotone paths. We have then found the shortest curve in the given homotopy class.



(a) Slanted rod bases have different y coordinates.



(b) Slanted rod bases have the same y coordinates.

Fig. 16: Cases for proof of Proposition 1.

C. Non-intersection of slanted rods

There is a slight technical problem that we might be concerned about, however. The apices of the parallel portions of the rods are the bases of the slanted portions of the rods. Arbitrary placements of the parallel rods might lead to slanted sections that intersect each other.

Fortunately, by careful design of the slanted sections, we can avoid this potential problem. Consider the top, slanted-rod section of the fixture. Before taking into account the need to move the parallel rods to eliminate occlusion, we might design this top section such that the fixture enforces a radial scaling of the goal configuration of the knot outwards to some less tight configuration. Since rays from a common point (formed by the scaling of the knot down to size zero) do not intersect unless they are coincident, there are no intersections between slanted rods using this approach. In fact, perturbations of these rays to avoid occlusion between their bases can also avoid intersection:

Proposition 1. *Given a set of points $p_i, i = 1, 2, \ldots, n$ in R^3 with the same z coordinates, there exist a set of points p'_i, $i = 1, 2, \ldots, n$ on the x-y plane where $x(p'_i) \neq x(p'_j)$ for $i \neq j$ such that the n line segments connecting each p_i to p'_i do not intersect each other.*

Proof. The set of points p'_i can be found in the following way. Let $z(p_i) = c$ for all i, where $z(*)$ represents the z coordinates of a point and c is a positive constant. Choose a value $h > 0$ representing the desired height of a radial projection center point above all p_i. Let this center point O have coordinates $(\frac{1}{n}\sum_{i=1}^{n} x(p_i), \frac{1}{n}\sum_{i=1}^{n} y(p_i), c+h)$. Then for any i, the point p''_i is on the ray Op_i with $z(p''_i) = 0$.

If $y(p''_i) \neq y(p''_j)$ for $i \neq j$, choose any p'_i and p'_j such that $y(p'_i) = y(p''_i)$ and $y(p'_j) = y(p''_j)$ (Figure 16a). Denote the plane $\mathbb{P}_k$ as the plane that contains the line $y = y(p''_k)$ and ray Op''_k for any k. Plane $\mathbb{P}_i$ and plane $\mathbb{P}_j$ intersect at the line that passes through O parallel to the x axis. We know $z(p_k) < z(O), k \in \{1, 2, \ldots, n\}$. Choose p'_i and p'_j to have the same y coordinates as p''_i and p''_j respectively, they belong to two different planes. Therefore, line segments $p_i p'_i$ and $p_j p'_j$ do not intersect.

If $y(p''_i) = y(p''_j)$, choose p'_i and p'_j such that $\text{sign}(x(p'_i) - x(p'_j)) = \text{sign}(x(p_i) - x(p_j))$ (Figure 16b), $p_i p'_i$ will not intersect $p_j p'_j$.

Overall, we can easily enforce $x(p'_i) \neq x(p'_j)$ for $i \neq j$; therefore we have found p'_i. ☐

Therefore, if we first radially scale all rods outwards from p''_i, then move to p'_i to guarantee visibility, the slanted rods connecting p'_i to p_i for all i will not intersect each other.

The proof assumes the rods are of zero radius, but if the cross sections of rods are circles with radius r, and $|y(p'_i) - y(p'_j)| < r$, the rods can still intersect. In this case, we can find O' where $z(p_i) < z(O') < z(O)$, such that the new p'''_i on the ray $O'p_i$ satisfying $|y(p'''_i) - y(p'''_j)| > r$. Using p'''_i to replace p''_i and finding new p'_i resolves the problem.

With the assurance that there always exists a design such that the slanted rods do not intersect, we designed such tightening system for shoelace unknot and double coin knot. The complete setup for tightening the shoelace unknot is shown in Figure 17.

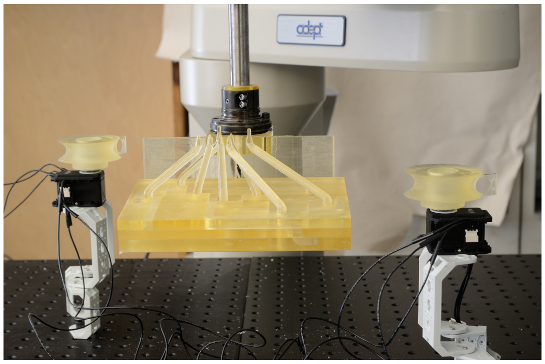
Figure 21 shows the fixture designed for tightening a double coin knot, and the result of a tightening.

D. Tightening fixtures with moving rods

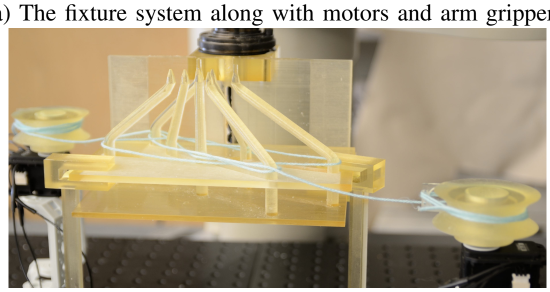
During tightening of the previously mentioned shoelace unknot in Section I-A and double coin knots, we observed several violations of our assumptions, even though the tightening was successful cross several repeated trials.

The string was not always taut around the rods during tightening, since friction between some rods and the string can prevent equal distribution of tension between segments of string. Friction also prevented the string from sliding vertically along rods at the same speeds, resulting in the situation shown in Figure 18. Since the design is based on the idea that the z axis may substitute for the time axis in space-time, we would like the string to always wrap around the rods at the same z coordinate throughout the tightening process. The difference in z coordinates was large in the experiments.

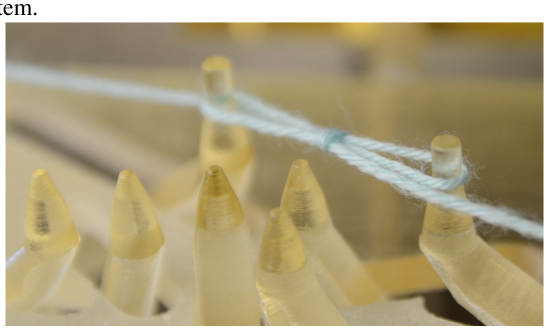
Different tilting angles of the rods can cause string to slide along the rods vertically with different speeds. Friction between string and rods can also cause string to catch during sliding.



(a) The fixture system along with motors and arm grippers.



(b) The starting configuration of the knot around the tightening system.



(c) The final configuration of the knot tightened using the system.

Fig. 17: The automation system for tightening, and the starting point for shoelace knot.

The capstan equation [7] tells us that when string wraps around rods in the presence of friction, the tension before contact with rods and after may be different. To reduce this type of friction, we considered an alternate design. In this design, the rods are on ball bearings, such that when the string is taut, the rods rotate with the string, leading to low effective friction in the horizontal plane containing the string.

Vertical friction along the rods is also a problem. Rather than using tilted rods, in the second fixture design, we placed a collection of straight rods on sliders that can move along tracks in the x-y plane passively when pulled on by string. The shape of the tracks is the projection of the tilted rods used in the first design. The straight rods move along the tracks with low friction.

The passive motion of the rods, powered by the string itself, is not perfect, since different forces may be applied to different rods, leading to non-uniform scaling of the knot shape during tightening. However, the simplicity of the passive design is attractive, and was sufficient to ensure correct tightening for the knots we tried (Figure 19). Among the knots we have experimented on, the non-uniform scaling appears not to be a problem. However, we expect for a larger and more complex knots, the non-uniform scaling will lead to premature friction-lock, preventing the knot from being tightened correctly.

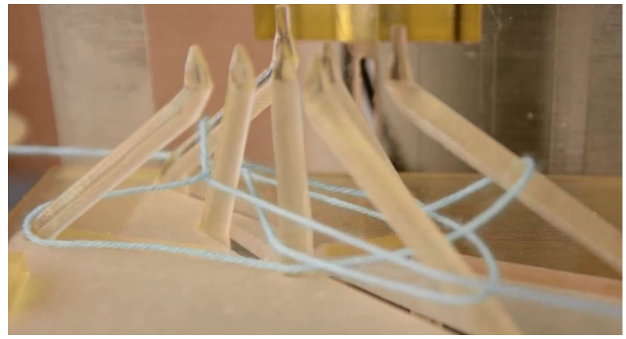
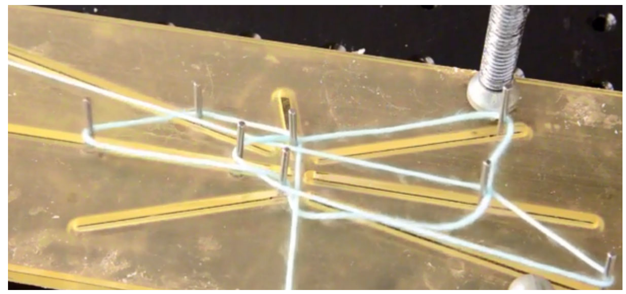
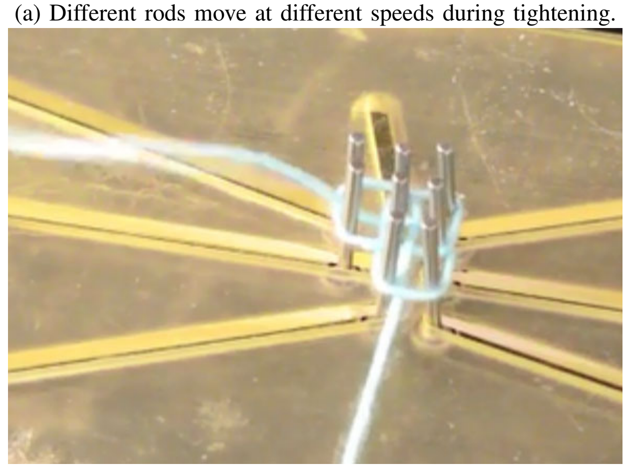


Fig. 18: During tightening shoelace unknot, the z coordinates are clearly different at different locations on the string.



(a) Different rods move at different speeds during tightening.

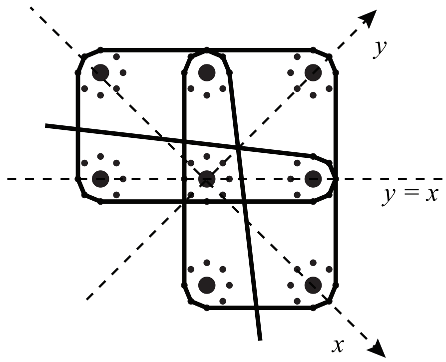


(b) The knot eventually reaches the desired geometry at the end of tightening process.

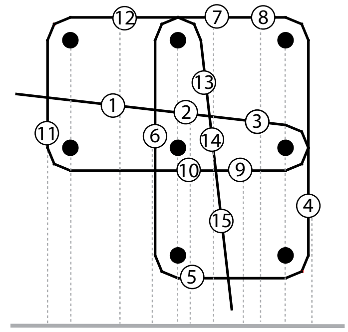
Fig. 19: A fully passive tightening fixture using tracks. The rods slide along the tracks when the string contacts the rods to provide force.

V. Automated design process

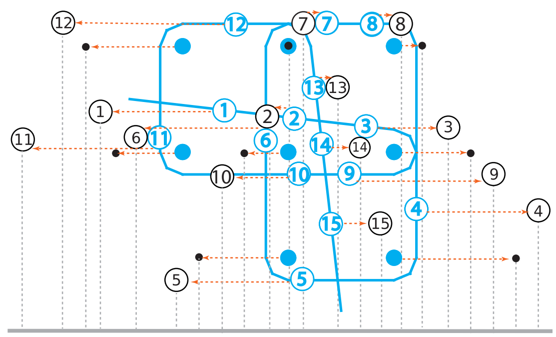
Previous sections explored aspects of the design of fixtures; this section describes the complete design process, first using



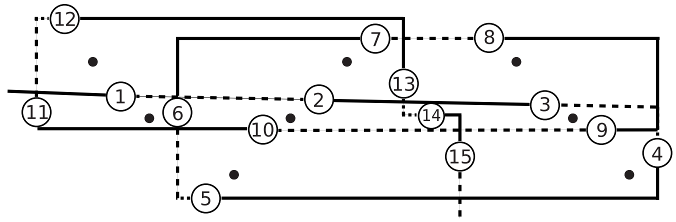
(a) Finding the shortest curve around points guaranteeing a minimum clearance from rods. Dashed lines show the original axis and the chosen $y = x$ axis on which the vertical sections will be projected.



(b) Based on the sequence of crossings, identify the vertical sections of string (circles on the string), and the rods. Occlusions are shown by projecting rods and vertical string to the chosen axis (gray line on the bottom) using dashed lines.



(c) Remove occlusions.



(d) Simplified layout of string around rod positions, using computed vertical segment positions from (c).

Fig. 20: The fixture design process.

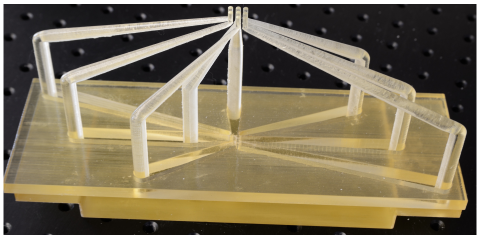
the double coin knot as an example, and then considering a more complex example, the cloverleaf knot. The design process is the reverse of the tying process: from the final desired configuration of the knot, arrange rods, scale up the knot, and embed knot and rods in an arrangement fixture, enforcing that all the vertical string segments and rods are visible from a chosen direction.

We start with a photograph of the knot in its desired configuration along with corresponding crossing sequence (knot diagram), such as the double coin knot in Figure 12a. By hand, we mark the outline of the knot shape in the photograph, roughly identifying the geometric configuration of the knot and the centers of the cells. The centers of the cells then form the top of the tightening fixture, as shown in Figure 12b. Admittedly, the digitalization process can be automated by identifying string from background using computer vision. However, this is not the focus of this paper, so we will not discuss the digitalization in more details.

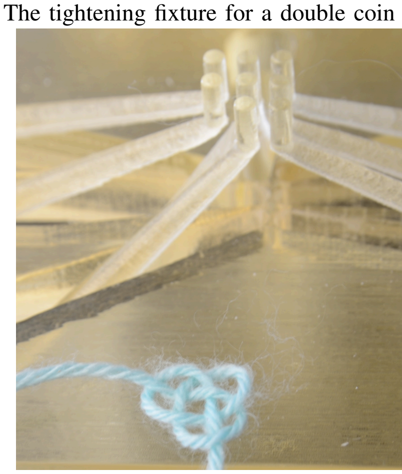
The next step is to choose an arbitrary direction along which to enforce the visibility property on all rods and vertical string elements. In Figure 20, we chose the original $y = x$ line to be the axis onto which vertical elements are projected to determine visibility. For convenience, we then transform the coordinate system so that the projection axis is the x-axis.

In the transformed coordinate system, find the shortest homotopy curve of the string among given rods. We chose to use eight points to approximate each disc, with some clearance to allow for the arrangement fixture to surround the rods. An example is shown in Figure 20a.

Along the shortest homotopy curve, based on the knot crossing information, identify all the vertical string segments (denoted as segment nodes in [11] [12]), as shown in Fig-



(a) The tightening fixture for a double coin knot.



(b) A double coin knot tightened using the fixture in Figure 21a.

Fig. 21: The tightening fixture for the double coin knot and a knot tightened using this fixture.

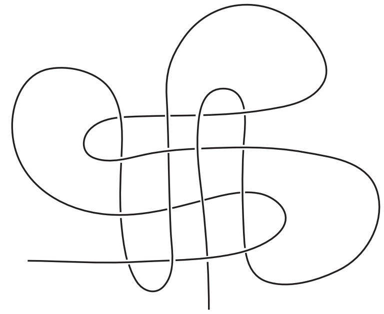
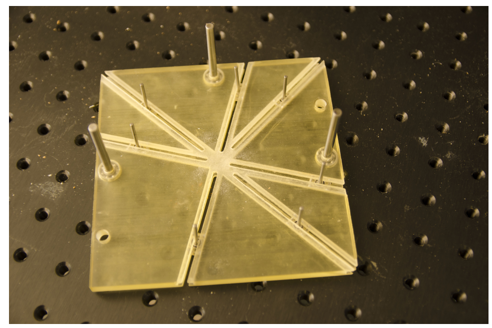
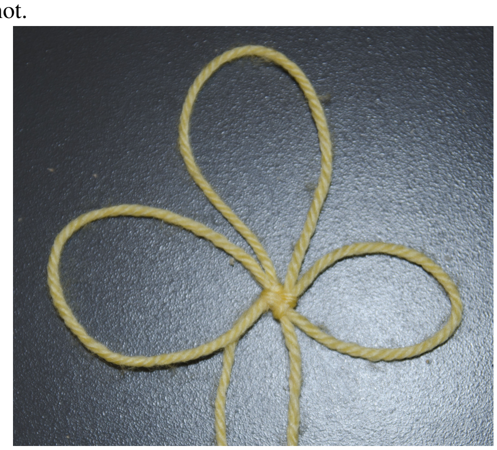


Fig. 22: Knot diagram of the cloverleaf knot.



(a) The tightening fixture with moving rods for cloverleaf knot.



(b) A cloverleaf knot tightened by our fixture.

Fig. 23: The designed tightening fixture for cloverleaf knot, and a tightened cloverleaf knot using the fixture.

ure 20b. (For convenience of the human designer, we may also orthogonalize the tubes for the string.)

Finally, remove occlusions using the procedure proposed in the proof of Proposition 1. Figure 20c shows the occlusion-removal procedure and Figure 20d shows the resulting layout of the tubes. Based on this layout, a human designer can model the arrangement and tightening fixture in 3D modeling software. (This step is required because the layout, although it contains all of the most interesting required information, is two-dimensional and not formatted for 3D printing.)

The resulting tightening fixture for a double coin knot is shown in Figure 21a. We applied our tightening approach using the printed fixture, and tied the double coin knot shown in Figure 21b, which is similar to the configuration shown in Figure 12a.

A. *Tightening the cloverleaf knot*

This section discusses an example of tightening the *cloverleaf knot* shown in Figure 22. This given knot is quite complex compared to previously presented knots. The knot has open 'ears' that need to be maintained, like the shoelace unknot. At the same time, the structure of the tightened knot is hard to maintain by simple pulling without any assistance. An example of what can go wrong when pulling on just the endpoints of the string by hand without any fixture is shown in Figure 2b. (The hand-tightened cloverleaf shown in Figure 2a required a complex and, in our case, ad-hoc, sequence of maneuvers to keep the loops open and correctly sized, and to keep the center well-structured.)

The designed fixture for tightening cloverleaf knot is shown in Figure 23a. We conducted experiments tightening the cloverleaf knot with consistent success over tens of trials; an example tightened knot is shown in Figure 23b.

We recorded the tightening time for different knots using fixture (after arrangement), and compared to tightening by hand (Table I). The tightening by hand was slow because of the need to carefully maintain the geometry of the knot by hand during tightening. Of course, we do not claim that our inexpert hand-tightening should be considered the gold standard for human knot-tying – these experiments were only

Method \ Knot type	Shoelace	Cloverleaf
Fixture (moving rods)	4.46 (0.23)	7.98 (0.94)
Fixture (slanted rods)	35.7 (1.23)	N/A
Hand	2.01 (0.23)	15.23 (0.80)

TABLE I: Time (in seconds) used during tightening for different knots, both using fixture or tightening by hand. Reported time is the average over 10 runs. The numbers in parentheses are the standard deviation.

intended to give a general idea of the speed of the current fixtures.

VI. Conclusions and future work

This paper showed a process to design fixtures to arrange and tighten knots and unknots. Examples included the double coin knot and shoelace unknot.

The mechanical implementation of the fixtures could be improved: less flexible rods with smoother surfaces for better sliding, a device for automatically disassembling the arrangement fixture, friction reduction inside the arrangement fixture using ball bearings, and better control of the geometry of the string during tightening. Actively controlled movable parts in the tightening fixture may allow an even more controlled tightening.

We would like to expand the set of knots that can be tied. Any knot that can be described by a knot diagram can be arranged by a sufficiently large arrangement fixture together with a feeding mechanism capable of inserting the string fully, but only flat knots can be tightened using the tightening fixture. How can more general knots be tightened? Can knots be tied around large structures, such as gift-wrap boxes, or tied quickly enough to be used in medical applications?

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