Towards tying knots precisely

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***Abstract*— We show that using a fixture-based approach, knots can be tightened satisfying fairly precise geometric constraints; for example, we might like the loops of tied shoelaces or a decorative cloverleaf knot to each be some precise length. A robot arm and specially designed gripper are used first to arrange knots on the fixture, comprised of a collection of straight rods. Some of the rods can move; precise tightening is achieved by pulling of the open ends of the string, while rods guide and delay friction locks appropriately. We show an algorithm for designing a fixture based on an arbitrary input knot diagram. Some example knots, including the cloverleaf knot, are tied as a proof of concept.**

I. Introduction

In this work, we present an approach to tying knots precisely without simulation or sensing. By precisely, we mean that when a knot is tight, specified distances will be achieved between selected pairs of contact points. These distance constraints are enforced by controlling the locations where friction locks are formed during tightening of the knot.

Knot-tying is worthy of study as a fundamental manipulation process. How do you manipulate flexible materials into configurations with certain qualitative or geometric properties? Although our current focus is on fundamental processes, there may ultimately be some practical use for systems that tie decorative or practical knots precisely, in areas ranging from automated manufacturing to surgical robotics.

Historically, sailors measured water depth using a *sounding line* with knots tied at six foot (one fathom) intervals along the line; sounding lines are one example where knots need to be placed precisely. Other example include decorative knots, such as the *cloverleaf knot* shown in Figure 2, tied by one of our fixtures (Figure 1).

Mathematical *knot diagrams* are typically used to describe the structure of a knot. A knot diagram is a regular projection of a loose knot (or unknot) onto a plane, where broken lines indicate one segment of the string under-crosses some other segment. Knot diagrams do not themselves indicate the precise geometry of the tightened knot; additional information about the distances along the string where points of contact between different parts of the string occur is needed.

One of our central ideas is to manipulate the string while it is tautly held by a fixture, so that deformation of the string is limited, allowing easy modeling and control. The first step of our approach is to *arrange* the knot on the fixture that will be used to *tighten* the knot. Although we have previously designed nearly passive *arrangement fixtures* [34], requiring only the application of pressurized air, the larger size and complexity of knots considered in this paper makes insertion of the string difficult using that method. Therefore, we use a robot arm with a special-purpose gripper to arrange the knot around the fixture.

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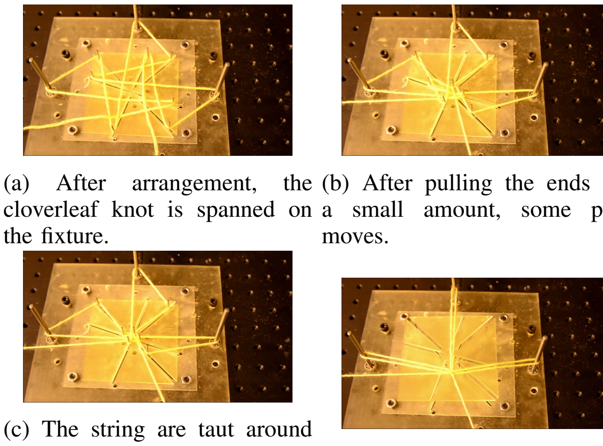


(a) After arrangement, the cloverleaf knot is spanned on the fixture.

(b) After pulling the ends for a small amount, some pins moves.

(c) The string are taut around the rods, cannot be pulled further.

(d) The cloverleaf knot is tight around the tightening fixture.

Fig. 1: Tightening a cloverleaf knot.

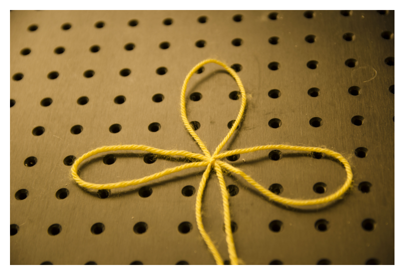


Fig. 2: Machine-tied cloverleaf knot.

The knot is then *tightened* on the fixture by pulling on the ends of the string, as shown in Figure 1. The fixture uses rods to span the knot during arrangement, and to delay the friction locks from happening prematurely. Some rods are removed during tightening, and others are allowed to move, leading corresponding friction locks to appropriate locations to satisfy the distance constraints. The majority of the paper introduces fixtures with rods that move passively due to the tension along the string during tightening. In Section VII, a motor is used to actively move these rods on the same fixture, allowing better control over the location of the rods.

To our knowledge, this is the first work on tying knots precisely with fixtures. The paper focus on the mechanical designs for this approach. The design of the fixture, however, can be automated using simple optimization techniques, and will be presented in Section V. Although this work is preliminary, and exhaustive experiments have not yet been performed, we show that a simple version of the sounding line and a decorative cloverleaf knot can be tightened precisely.

A. Related work

We are not aware of work that explores tying complex knots precisely in a general way. The use of fixtures to tie knots has been explored in our previous work [10], [34], both for arrangement and for tightening. Recently, fixtures have been used in cancer treatment [27] with paths designed to guide ribbon cable to a desired location without excessive bending or twisting. The rest of this section provides a short survey on knot tying; a lengthier survey can be found in [9].

Knot theory has long been studied in mathematical community [1], [2], [22], [30], including work on *knot invariants* which study different drawings of the same knot.

When is a knot tight? This question has been studied in the applied mathematics and physics [28], [8], [24] communities. It is useful to measure distances along the string in units of string thickness; such a measure is referred to as *rope length* in [5].

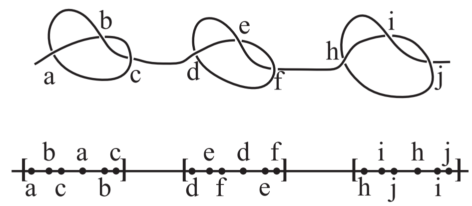
Special machines that can tie specific knots have been invented [13], [14], [31], [32], but the approaches are quite specific to the particular knots being tied.

Various approaches have been used to achieve particular string manipulation tasks. For example, wires are modeled as elastic rods for manipulation [33], and flexible needles can be used to navigate string [4], [3]. String manipulation has also been used as a challenge problem to study single- and multi-arm coordination [20], [21] .

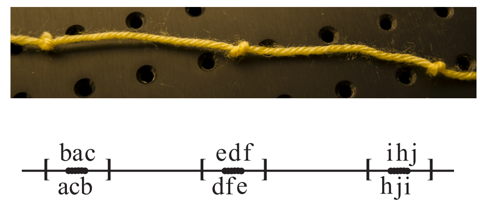
One critical observations is that when pulled taut around rods on tightening fixture, the string can be modeled and controlled much more easily. The taut configuration of string among rods has been studied in computational geometry community as the shortest curves in a particular homotopy class among point obstacles [12], [15], [17], [18], [19].

Friction exists between the contacts of string and the fixture, and even between the contacts of string and string. Rather than relying on analysis of these frictional forces (as in, for example, [7], [25], [11]) we mitigate friction by mounting all rods on low friction ball bearings, following the direction explored by Furst and Goldberg [16]. Motion planning studies of flexible bodies' minimum energy configurations [26], [29] are another source of inspiration leading us to manipulate the string when it is taut.

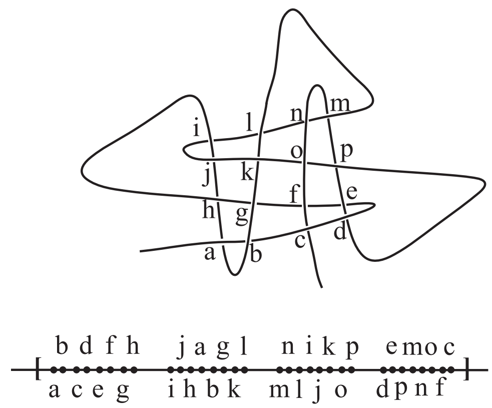
Manipulation of cartons by Lu and Akella [23] also inspired our current work, as has work on robotic folding of origami [35], [6].



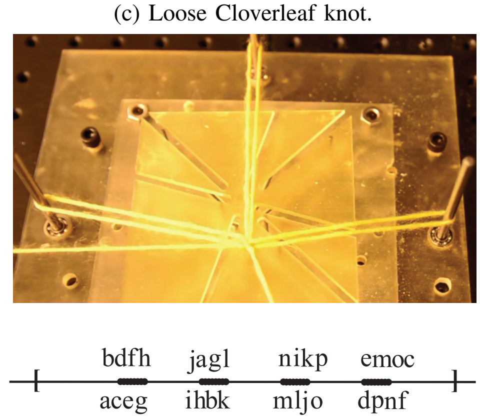
(a) Loose sounding line.



(b) Tight sounding line.



(c) Loose Cloverleaf knot.



(d) Tight cloverleaf knot.

Fig. 3: Contact diagrams on loose knot diagrams and machine-tightened sounding line and cloverleaf knot. The range between the brackets indicates the knot units, and the black circles indicates the crossings and their corresponding locations along the string.

II. Background: Precise Tightening

We consider the sounding line (top of Figure 3b) to be a *compound knot* in the sense that it is a collection of *knot units*. Here, a knot unit is defined as a segment of string $[s_i, s_j]$ that it is not in contact with any other piece of string.

To describe the distance constraints on a knot, we introduce the *contact diagram*, which is shown on the bottom of every subfigure in Figure 3. A contact diagram marks the distances of all crossings along the string, with matching square brackets indicating a knot unit. Precise tightening can then be described as the transformation of a knot from a loose state such as that in Figure 3a or 3c to a goal state: Figure 3b or 3d.

It is difficult to control friction locks once they have happened, so our goal is to prevent the friction locks from happening until they are all "in position". Straight rods are used to delay the friction locks. These rods on the tightening fixture are inserted into different *cells* defined by the knot diagram. A cell is an open bounded connected region of $\mathbb{R}^2$ enclosed by the knot diagram, which is a subset of the complement of the knot diagram [34].

We control the knot by controlling the motion of the rods during tightening. When the appropriate cells reduce in size, the crossings along the boundaries of cells move closer. At the goal state, the cells are reduced to minimum sizes at specified locations, causing selected crossings to be located at appropriate distances from each other.

III. Arrangement

In previous work [10], [34], we showed that theoretically, fixtures can be used to arrange any knot. However, fixtures designed in this way for the knots we consider in this work are too large to allow easy threading of string of string through the fixture. In this work, we use a robot arm to arrange knots on tightening fixtures of a different design.

Arranging a knot on the tightening fixture presents several challenges, including how to achieve the under crossings, and how to overcome the friction of dragging the string through the rods on the tightening fixture.

To overcome the friction, we used a spool to lay out the knot instead of dragging the string through the rods on the fixture. The spool is mounted on a ball bearing, so the string can be laid out during arrangement with minimum friction as the robot arm moves the spool around the tightening fixture. The tension along the string keeps the string wrapped around the rods at certain height.

To arrange under-crossings, we used a specially designed gripper that can re-grasp without dropping the spool. Figure 4 shows the gripper configurations for performing re-grasping. The gripper uses two linear motors, with an electromagnet attached to the tip of each linear motor. The spool is attached to a slider. There are two metal plates on top of the slider, allowing the gripper to release or grasp the slider using the electromagnets.

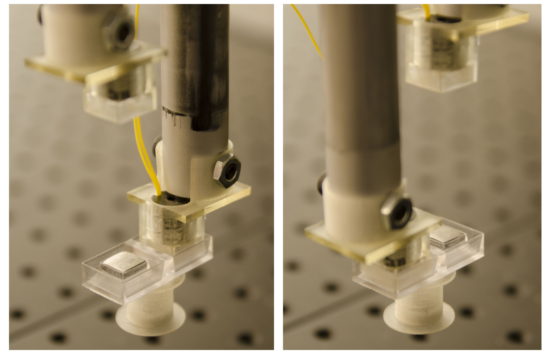


Fig. 4: Two different configurations of the gripper grasping the spool used to arrange the knots.

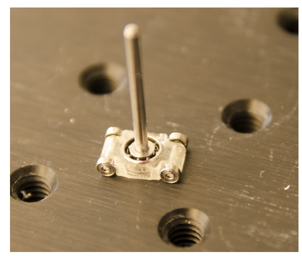


Fig. 5: The slider used for moving rods. The base has four wheels that are made of small ball bearings, while the rod is also on a ball bearing.

Using this gripper, sliders can be passed under segments of string with only one motor attached and be picked up by the other motor. The robot arm simply follows the knot diagram to lay out the knot on the fixture, performing re-grasps as needed. An example arrangement of the sounding line can be seen in Figure 8.

IV. Tightening

The tightening fixture contains a collection of straight rods, some of which can move along certain line segments, which guide the rods to predefined locations where selected cells will reach their minimum sizes.

In order for the rods to move, they are mounted onto sliders, as shown in Figure 5. Each slider is mounted on four small ball bearings that allow it to ride along a track inset in the fixture; an algorithm for automatic design of placement of these tracks will be described in Section V. The rod is also mounted onto a ball bearing, so that the rod can rotate to reduce friction during tightening when pulled.

We will first consider a simple tightening fixture, in which the rods are moved by the tension along the string. During tightening, two motors pull the open ends of the knot, tightening the string and causing sliders to move along tracks.

Certain stationary rods are used to keep the size of certain cells. In cloverleaf knot, for example, the distance constraint between crossing pairs are enforced by three stationary rods to maintain the perimeters of selected cells.

After the rods stops moving, the knot is almost tight and lifted away the moving rods to be tightened further, forming tight friction locks. This last step, however, is not controlled, and this is a source of error in the tightening process.

The tightening process of a cloverleaf knot is shown in Figure 1. The motors each pull the string by predefined length to lead the moving rods to their desired locations. These lengths are calculated using the difference in rope length between loose and tight configurations of the knot.

V. Fixture Designs

In this section, we will show that the tightening fixture can be designed nearly automatically for a given input knot diagram, based on some sequential design process. Although we have implemented code for each of the steps in Procedures 1 through 3, some of the steps are heuristic in nature, and must be manually checked by the designer; we will point out such steps in the discussion below.

The tightening fixture serves two purposes. First, it spans the knot during the arrangement, allowing the sequence of over and under crossings to be arranged. Second, after the arrangement, the fixture is used to guide the knot to a nearly tight configuration.

We use a knot diagram as a starting point to enforce the correct topology of the knot.

In previous work [34], which did not focus on precise tightening and only considered smaller scale knots, we simply inserted one rod per cell of the knot diagram. However, for simplicity of mechanical design, we would like to avoid using the implied number of 16 rods for the cloverleaf knot.

One key observation is that when the string is pulled tight around all inserted rods, many rods are not in contact with the string. Procedure RodPlacement first identifies the cells, putting a rod in each cell. We then find the shortest curve among these rods, using algorithm for finding the shortest curve of a particular homotopy class among points [12]. (We sample points along the boundaries of discs representing rods, as in [34].) We finally remove rods not involved in this curve. For the cloverleaf knot, this reduced the number of rods on the tightening fixture to 10.

We also would like to avoid letting multiple strands of string contact the same rod, which would lead to undesired motion during tightening due to friction as two strands of string apply different torques to the rod, preventing free rotation. We separate the contacting strands of string by doubling any such rods; an example is shown in Figure 6. Although we have implemented code for placement of the double rods, our method is heuristic, and must be manually checked to see if it introduces topological or geometric issues.

The current fixture can be used to span the knot during arrangement, but some places might be too small for the

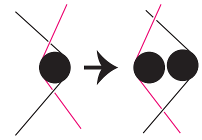
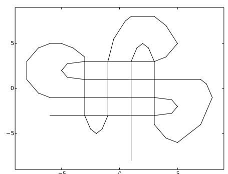
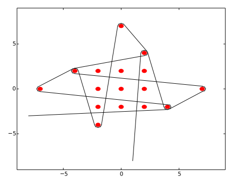


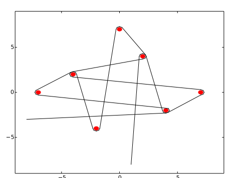
Fig. 6: When multiple strands of string contact the same rod, double the rod.



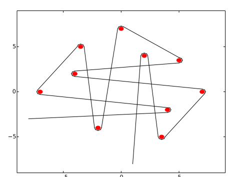
(a) The polygonal approximation of the cloverleaf knot diagram.



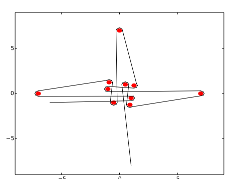
(b) Place a rod in each cell.



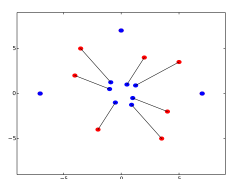
(c) Delete all rods that do not cause cell degeneration.



(d) Split rods to let each rod contact only one strand of string.



(e) Reduce the rope length. Result after Procedure ReduceRopeLength.



(f) Design the tracks by connecting corresponding rods using a line segment.

Fig. 7: Automated design of the fixture for the cloverleaf knot using Procedures RodPlacement to ReduceRopeLength.

gripper designed to perform the re-grasp task. We can add certain size constraints to each cell where a re-grasp might happen. This step (Procedure IncreaseCellSizes) happens after Procedure RodPlacement.

After all the rods are placed, we find a set of line segments (tracks) that guide the moving rods to a nearly tight configuration using Procedure ReduceRopeLength.

Finally, it is important to control the distance between contact points, for precise tightening. The distance is calculated between two nearest crossings. If the distance is not correct on the calculated curve wrapped around the rods, then we move the corresponding static rod parallel to the direction of the normal formed with the string to meet the distance

Procedure RodPlacement

Input: polygonal drawing $\mathscr{D}$; cells of $\mathscr{D} : \mathscr{C}_i$;
for *each cell* $\mathscr{C}_i$ **do**
  Find the largest containing disc with center d_i and radius r_i;
$r \leftarrow \min_{\forall i} r_i$;
Place a disc of radius r at each d_i;
Find the shortest homotopy curve among these discs;
for *disc centered at* d_i **do**
  remove the disc, detect if any cell degenerates;
  if *cell degenerates* **then**
    place disc with radius r centered at d_i;
while $\exists$ *disc at* d_i *contacting two stands of string* **do**
  double the disc so each disc only contacts one strand of string;

Procedure IncreaseCellSizes

Input: Crossing sequence: $\mathbb{S} = \{s_1, s_2, \ldots, s_n\}$, mapping $f(s_i) : \mathbb{N} \rightarrow \mathbb{R}^2$
for *undercrossing* s_i *and next overcrossing* s_j **do**
  if *distance between* $f(s_i)$ *and* $f(s_j)$ *too small* **then**
    Find the nearest rod, double the rod to increase the distance between $f(s_i)$ and $f(s_j)$;
    set s_i to be the under crossing after s_j;

constraint.

We applied the algorithm to design the tightening fixture for a cloverleaf knot. Figure 7 shows the important steps of the fixture design. We printed the fixture using the result of the design, after some work with a 3D modeling package, Solidworks, to model tracks and create sockets for static rods.

VI. Physical demonstration of precise knot-tying

We will use the examples of tying a sounding line and the cloverleaf knot to show the approach. An Adept Cobra Robot Arm was used for arrangement, and all tightening fixtures were prototyped using OBJET Eden 3D prototyping machine.

A. Sounding line

We first tied a simple sounding line with three knot units. In Figure 8, we show the arrangement, focusing on the re-grasping task during the arrangement of the first knot unit. We fixed one end of the sounding line for arrangement and tightening.

After arrangement, the open end is attached to a servo-motor, and the robot arm is used to remove the spanning pins. The knot then is tightened around the third pin. Since the knot unit is very simple, it was not necessary to use any

Procedure ReduceRopeLength

Input: rope thickness $\mathrm{Thi}(\gamma)$; small clearance a, geometric center of all rods c; a set of stationary rods D
Output: Tight ropelength Lp; disc locations $\mathscr{D}$
for *disc* $d_i \notin D$ **do**
  connect d_i and c;
  $k \leftarrow$ number of intersections between $\vec{d_i d_j}$ with the knot drawing;
  move d_i along $\vec{d_i c}$ to d_i' such that $d(d_i', c) = k \cdot \mathrm{Thi}(\gamma) + a$;
  $d_i \leftarrow d_i'$;
Calculate distance between contact points;
Move rods if distance constraint is not met;
Calculate ropelength $\mathrm{Rop}(\gamma)$;
$Lp \leftarrow \mathrm{Rop}(\gamma)$;

moving rods. Figure 10 shows the first knot unit tightened around the stationary pin after the two spanning pins are pulled out.

The distances between the static rods (78 mm each) were intended to match the distances along the string when tied, and when the string is taut, this is the case. However, the yarn we use in the experiment is quite stretchy, so the problem of precisely specifying the knot geometry is not entirely well-defined. When removed from the fixture, lengths measured along the loose string were 72 and 77 mm. We believe the difference in these two distances to be largely caused by different tensions on the section of the string introduced during tightening. A major goal of future work is to gain a better understanding of the relationship between lengths in loose and tight string, and to conduct rigorous experiments determining the precision that can be achieved.

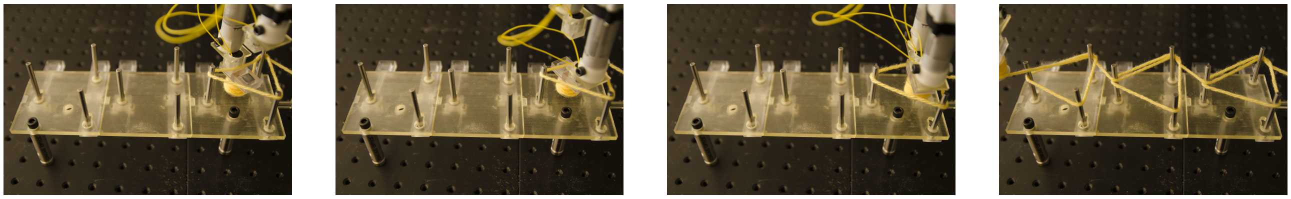
B. Cloverleaf knot

The arrangement for cloverleaf knot, though more complex, also relies on the success of the re-grasp tasks. The arrangement sequence is omitted in the paper, but can be seen in the video attachment.

After the arrangement, the knot is tightened by pulling the open ends. The moving pins approach desired locations. The sequence is shown in Figure 1. A machine-tightened cloverleaf knot is shown in Figure 2. Lengths of the three large loops in the knot measured along the fixture were 205 mm. The lengths of the sections of loose string after removal were 189 mm, 190 mm, and 191 mm.

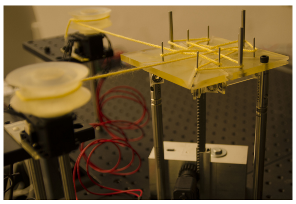
VII. Active tightening fixture

In the demonstration of tightening the cloverleaf knot, we observed that some rods did not reach the desired locations, perhaps due to the inconsistent direction between the string contact normals (along which we expect force to be applied

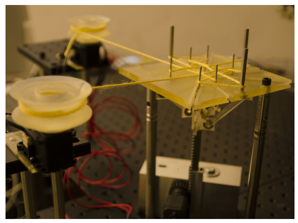


(a) Before re-grasp task in the first unit. (b) After the re-grasp task in the first unit. (c) About to finish arranging for the first unit. (d) Finished arranging three units of sounding line.

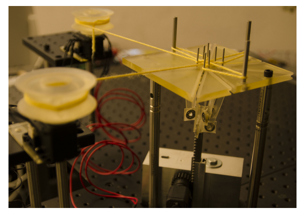
Fig. 8: sounding line arrangement process.



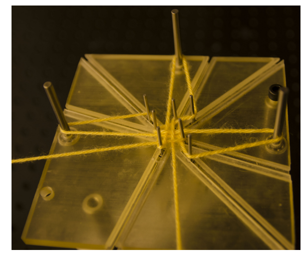
(a) Start of tightening with active fixture.



(b) During tightening with active fixture.



(c) Tight around active fixture.



(d) Tightening with active fixture; top view.

Fig. 9: Tightening cloverleaf knot using active fixture.

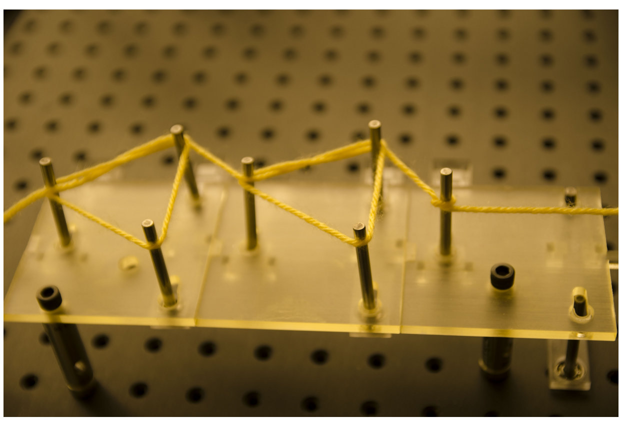
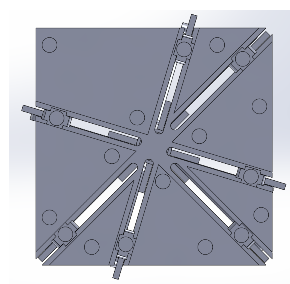
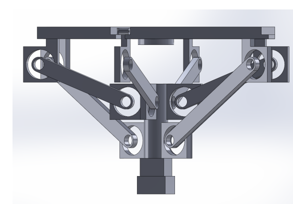


Fig. 10: The first knot unit along the sounding line is tightened around the stationary pin.



(a) Top view of tracks for a cloverleaf knot.



(b) Side view of the bar-linkage system. There are two different link lengths, to move different rods with different speeds.

Fig. 11: Model of the track and bar-linkage system for a cloverleaf knot.

to the moving pin) and the directions of the tracks. We therefore attached all sliders to a single motor, using a bar-linkage system; an example for the cloverleaf knot is shown in Figure 11.

We conducted the tightening task again for the cloverleaf knot using this active fixture, allowing all rods to reach their desired locations, as shown in Figure 9.

VIII. Conclusions, Future work and acknowledgments

In this work, we demonstrated some fixtures designed for tying knots precisely. The main idea is to delay the friction lock, while moving all desired contact points into position. Before tightening, a robot arm is used to arrange the knot using a specially designed gripper. This approach is quite general; we presented a design procedure that can be used for essentially arbitrary planar knot geometries.

Using the fixtures, we tied a sounding line and cloverleaf knot fairly precisely. Obvious sources of error include the stretch of the string and the fact that the final tightening motion (once desired contacts have nearly been reached) is not precisely controlled.

Our next aspirational goal is to tie a *Ruyi knot* precisely. The compound Ruyi knot contains four cloverleaf knots, three of them embedded into the fourth knot unit; see Figure 12. A major challenge for this work will that pulling

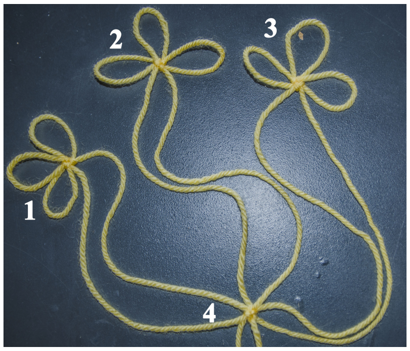


Fig. 12: A hand tied Ruyi knot.

only from the ends does not appear to pull string through the fixture, even with ball bearings allowing rods to spin freely.

Most directly related to the current work, we intend to move from simple proofs of concept to rigorous experimental work on a wide variety of knot types.

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References

[1] C.C. Adams. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. American Mathematical Society, 2004.

[2] J. W. Aexander. Topological invariants of knots and links. *Trans. Amer. Math. Soc.*, 20:275–306, 1923.

[3] Ron Alterovitz, Ken Goldberg, Jean Pouliot, Richard Tascherau, and I-Chow Hsu. Sensorless planning for medical needle insertion procedures. In *IROS*, pages 3337–3343. IEEE, 2003.

[4] Ron Alterovitz, Andrew Lim, Kenneth Y. Goldberg, Gregory S. Chirikjian, and Allison M. Okamura. Steering flexible needles under markov motion uncertainty. In *IROS*, pages 1570–1575. IEEE, 2005.

[5] Ted Ashton, Jason Cantarella, Michael Piatek, and Eric Rawdon. Knot tightening by constrained gradient descent. *Experimental Mathematics*, 20(1):57–90, 2011.

[6] Devin J. Balkcom and Matthew T. Mason. Robotic origami folding. *I. J. Robotic Res.*, 27(5):613–627, 2008.

[7] Devin J. Balkcom, Jeffrey C. Trinkle, and E. J. Gottlieb. Computing wrench cones for planar contact tasks. In *ICRA*, pages 869–875. IEEE, 2002.

[8] J. Baranska, S. Przybyl, and P. Pieranski. Curvature and torsion of the tight closed trefoil knot. *The European Physical Journal B - Condensed Matter and Complex Systems*, 66(4):547–556, 2008.

[9] Matthew P. Bell. Flexible Object Manipulation. Technical Report TR2010-663, Dartmouth College, Computer Science, Hanover, NH, February 2010.

[10] Matthew P. Bell, Weifu Wang (co-first author), Jordan Kunzika, and Devin Balkcom. Knot-tying with four-piece fixtures. *International Journal of Robotics Research (IJRR)*, vol 33, no. 11:1481–1489, Sep, 2014.

[11] Stephen Berard, Kevin Egan, and Jeffrey C. Trinkle. Contact modes and complementary cones. In *ICRA*, pages 5280–5286, 2004.

[12] Sergei Bespamyatnikh. Computing homotopic shortest paths in the plane. In *SODA*, pages 609–617. ACM/SIAM, 2003.

[13] James Burns and Andrew Fung. Shoelace knot assisting device, May 2006.

[14] Mark Champion. Knot tying device, November 2004. US Patent 6817634.

[15] Alon Efrat, Stephen G. Kobourov, and Anna Lubiw. Computing homotopic shortest paths efficiently. *Comput. Geom.*, 35(3):162–172, 2006.

[16] M.L. Furst and K.Y. Goldberg. Low friction gripper, March 24 1992. US Patent 5,098,145.

[17] Dima Grigoriev and Anatol Slissenko. Computing minimum-link path in a homotopy class amidst semi-algebraic obstacles in the plane. In Teo Mora and Harold F. Mattson, editors, *AAECC*, volume 1255 of *Lecture Notes in Computer Science*, pages 114–129. Springer, 1997.

[18] Dima Grigoriev and Anatol Slissenko. Polytime algorithm for the shortest path in a homotopy class amidst semi-algebraic obstacles in the plane. In Volker Weispfenning and Barry M. Trager, editors, *ISSAC*, pages 17–24. ACM, 1998.

[19] John Hershberger and Jack Snoeyink. Computing minimum length paths of a given homotopy class. *Comput. Geom.*, 4:63–97, 1994.

[20] H. Inoue and M. Inaba. Hand-eye coordination in rope handling. *Robotics Research: The first International Symposium*, pages 163–174, 1985.

[21] Makoto Kudo, Yasuo Nasu, Kazuhisa Mitobe, and Branislav Borovac. Multi-arm robot control system for manipulation of flexible materials in sewing operation. *Mechatronics*, 10(3):371 – 402, 2000.

[22] W.B.R. Lickorish. *An Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer New York, 1997.

[23] Liang Lu and Srinivas Akella. Folding cartons with fixtures: A motion planning approach. In *IEEE International Conference on Robotics and Automation*, pages 1570–1576, 1999.

[24] J.H.Maddocks M. Carlen, B. Laurie and J. Smutny. Biarcs, global radius of curvature, and the computation of ideal knot shapes. *Physical and numerical models in knot theory*, 36 of Ser. Knots Everything:75–108, 2005.

[25] Matthew T. Mason. *Mechanics of Robotic Manipulation*. MIT Press, Cambridge, MA, August 2001.

[26] Mark Moll and Lydia E. Kavraki. Path planning for minimal energy curves of constant length. In *ICRA*, pages 2826–2831, 2004.

[27] Sachin Patil, Jia Pan, Pieter Abbeel, and Ken Goldberg. Planning curvature and torsion constrained ribbons in 3d with application to intracavitary brachytherapy. In *Workshop on the Algorithmic Foundations of Robotics (WAFR)*, August, 2014.

[28] Eric J. Rawdon. Approximating the thickness of a knot. *Ideal knots*, 19 of Ser. Knots Everything:143–150, 1998.

[29] Samuel Rodríguez, Jyh-Ming Lien, and Nancy M. Amato. Planning motion in completely deformable environments. In *ICRA*, pages 2466–2471. IEEE, 2006.

[30] D. Rolfsen. *Knots and Links*. AMS/Chelsea Publication Series. AMS Chelsea Pub., 1976.

[31] Wamis Singhatat. Intracorporeal knot tier, April 2004.

[32] Andrew Stone. Shoe tying robot. `http://www.youtube.com/watch?v=XrA7DR0u0uI`.

[33] Zoe McCarthy Timothy W Bretl. Quasi-static manipulation of a Kirchhoff elastic rod based on a geometric analysis of equilibrium configurations. *International Journal of Robotics Research (IJRR)*, June 2013.

[34] Weifu Wang, Matthew P. Bell, and Devin Balkcom. Towards arranging and tightening knots and unknots with fixtures. In *Workshop on the Algorithmic Foundations of Robotics (WAFR)*, August, 2014.

[35] Guowu Wei and J.S. Dai. Geometry and kinematic analysis of an origami-evolved mechanism based on artmimetics. In *Reconfigurable Mechanisms and Robots, 2009. ReMAR 2009. ASME/IFToMM International Conference on*, pages 450–455, June 2009.