Article 1

THE WALL STREET JOURNAL.

SCIENCE JOURNAL

As the Stakes Increase, Prime-Number Theory Moves Closer to Proof

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932 words
8 April 2005
The Wall Street Journal
B1
English

THE ENGLISH mathematician G.H. Hardy (1877-1947) was an avowed atheist, but not above hedging his bets. Whenever he had to cross the Channel, he mailed postcards to friends saying he had proved the "Riemann hypothesis," an intriguing mathematical conjecture about prime numbers that had been proposed (but not proved) by Bernhard Riemann in 1859. By the early 20th century the Riemann hypothesis had become a Holy Grail for mathematicians. Hardy was therefore sure that if, on the off chance, God did exist, He would never let Hardy take the proof, unpublished, to a watery grave (Hardy also was apparently sure God would fall for the empty boast on the postcard).

In the decades since, the legend of the Riemann hypothesis has only grown, becoming "the most important unsolved problem in mathematics," says mathematician Dan **Rockmore** of Dartmouth College, author of a nifty new book, "Stalking the Riemann Hypothesis: The Quest to Find the Hidden Law of Prime Numbers." Since 2000, the problem also has had a bounty on its head. The Clay Mathematics Institute, a private group in Cambridge, Mass., is offering $1 million to the first person who can prove it.

The prize sits unclaimed. But after a century of progress that can charitably be described as fitful, "frustration has begun to give way to excitement, for the pursuit of the Riemann hypothesis has begun to reveal astounding connections among nuclear physics, chaos and number theory," Prof. **Rockmore** says.

WHAT THESE appear to have in common is prime numbers, because deep down the Riemann hypothesis describes in detail how prime numbers are sprinkled along the number line. Primes are numbers that can be evenly divided only by themselves and 1. So 3, 5, 7, 11 and 13 are prime, as are 199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089.

This last string is curious because the primes in it all are separated by 210. Last spring, two mathematicians proved that there exist strings (separated not by 210 but by other intervals) that contain an arbitrarily-long run of primes. That is, you can find a number, keep adding another number to it and get a run of primes as long as you like. Because prime numbers underlie digital cryptography and Internet security, such deep truths have become more than mere oddities.

An early discovery about the primes was that there is an infinite number of them, sprinkled "like indivisible stars scattered without end throughout a boundless numerical universe," Prof. **Rockmore** writes. But how infinite? Although most of us think of infinity as one big number, some infinities are bigger than others. The number of numbers divisible by 2 is infinite, and so is the number divisible by 9. But the first infinity is bigger. There also is an infinite number of squares (4, 9, 16 . . .) and cubes (8, 27, 64 . . .), but more primes than either.

In 1859, Riemann got an inkling of how the primes thin out as you go along the number line. The number of primes around a particular number, he knew, equals the reciprocal of (that is, 1 divided by) the natural logarithm of that number. The natural logarithm of a number equals how many times you have to multiply a number called e (about 2.718) by itself to get that number. At around one million, whose logarithm is about 13, every 13th number or so is prime. At one billion, whose log is about 21, about every 21st number is prime.

RIEMANN WANTED to fathom why the heck primes were related to logarithms. He suspected he might find a clue in a formula that adds up 1 + 1/2 + 1/3 + 1/4 + 1-over-every-other-counting-number, but with the twist that each fraction is raised to an exponent (multiplied by itself some number of times). For bizarre exponents -- those that use the imaginary number the square root of -1 -- this sum equals zero. Riemann guessed at the general form of these "magical exponents." If his hypothesis is right, then mathematicians will know how primes thin out along the number line.

Proving the hypothesis means proving that every exponent of the form Riemann described makes the sum of the fractions zero. For more than a century mathematicians have been testing the magical exponents. In 1903 a researcher checked the first 15. By the 1930s, others had verified the first 1,000. By 1968, they had 3.5 million. Two years ago an IBM researcher using 500 computers verified the first 50 billion.

But that doesn't count as proving all of Riemann's exponents work. What if the 50-billionth-and-1st doesn't? The $1 million still is up for grabs.

The stakes are actually higher. The Riemann hypothesis now has been shown to underlie a plethora of puzzles in physics and math. The pattern of his magical exponents is related to the energies of particles in atomic nuclei, the energies of waves that fit precisely on geometric surfaces that describe space in Einstein's general theory of relativity, waiting times on bank lines and even how many cards you have to move to order the hand you're dealt in bridge. Why that should be so is -- depending how you look at it -- a coincidence, a profound truth of nature, or proof that God has a sense of humor. Maybe Hardy had the right idea with those postcards.

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