Numerical Investigation of the Spectrum for Certain Families of Cayley Graphs

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Abstract

In this paper we extend some earlier computations [8]. In particular, the expanding behavior of Cayley graphs of $PSL_2(F_{107})$ is compared with that of the Cayley graphs for the group A_{10}. These computations support the (up to now) unvoiced conjecture of Lubotzky that the symmetric groups and projective linear groups have asymptotically different average expanding behavior. We also give a thorough spectral analysis for a natural family of Cayley graphs which does not admit analysis by Selberg's theorem.

1 Introduction

Spectral analysis and operator theory have provided some of the main tools for the recent advances in constructions of expander graphs. In particular, by exploiting the various relationships between the second largest eigenvalue of the Laplacian and the expansion coefficient of graphs, families of expanders have been constructed and analyzed. When the graphs of interest are Cayley graphs, techniques from Fourier analysis are especially useful in this analysis.

In this paper we extend the analysis of [8] by presenting two different computations. The first computation gives an analysis of the spectrum of the Cayley graphs of $SL_2(F_p)$ on the generating sets $\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\}$. Viewed as elements of $SL_2(Z)$, the former set generates a subgroup of finite index, while the latter set generates a subgroup of *infinite* index in $SL_2(Z)$. Consequently, for the latter pair, the often useful techniques of Selberg's theorem are of no help in studying the asymptotic spectral behavior of the graphs. Our results in this first computation give further evidence for speculations presented in our earlier work (cf. [8], Section 6).

The second computation compares the average expanding behavior of Cayley graphs of $PSL_2(F_{107})$ with that of A_{10}. Our motivation for undertaking this computation again comes from our earlier work which served as initial evidence towards the conjecture of Lubotzky that the symmetric groups and special linear groups have asymptotically different average expanding behavior. It is well-known that if a pair of elements of $SL_2(F_p)$ is chosen at random, then with high probability (approaching 1 as p goes to infinity, at a rate of roughly $1 - O(\log^2 p/p)$) these elements will generate $SL_2(F_p)$ [6]. In [8] this led us to consider the following question: if two elements a, b are chosen uniformly from $SL_2(F_p)$ subject to the constraint that $\langle a, b\rangle = SL_2(F_p)$, then what can be said about $\lambda_1(SL_2(F_p), \{a, b\})$?[1] Our considerations were from a computational point of view, and letting p vary from 3 to 199 we chose 50 generating pairs at random, and computed λ_1 for each such pair. The results showed a marked clustering away from 1, roughly between 0.865 and 0.890. (For details as well as the graph, see [8], Section 5.3.) The clustering of the these eigenvalues naturally suggests that one should be able to prove that *asymptotically, almost every generating pair of $SL_2(F_p)$ is a good expander* ([8], Section 6, Question 4).

In a similar vein is an open question concerning the expanding behavior of the symmetric groups. In [9] Lubotzky asks the question, *can the family of symmetric groups be made into a family of expanders for any set of generators?*. Recent work seems to indicate that the answer to this question may be no. So, it was suggested to us by Lubotzky that we consider computing λ_1 for *every* pair of generators for comparably sized symmetric and special linear groups. The possibility of doing this is, of course, dictated by the size of the groups involved. But our experience shows that the spectral behavior of the special linear groups becomes "typical" for p near 100. Our techniques compute the spectrum by computing the appropriate Fourier transforms at a complete set of irreducible representations for $SL_2(F_p)$ (cf. Section 2). These representations have size on the order of p, so that the eigenvalue computations pose no problems for $p \approx 107$.

For the symmetric groups, our understanding of the asymptotics of numerical spectral analysis is less well-developed. A serious computational consideration for the symmetric groups is that the degrees of the irreducible representations grow quickly [7]. Thus, the determination of the eigenvalues soon dominate the computation. The symmetric group S_{10} is still computationally tractable from this point of view (with largest irreducible representation of degree 768). Its size is comparable with $SL_2(F_p)$ for $p = 107$. We simplify the problem of computing the spectrum of generating pairs by considering only the 3-regular Cayley graphs for $PSL_2(F_{107})$ and A_{10}, taking one generating element to be an involution.

The remainder of this paper is organized as follows. In Section 2 we review the relevance of Fourier analysis for spectral computations of Cayley graphs. In the Section 3 we present a very brief description of the representation theory of the groups $SL_2(F_p)$, $PSL_2(F_p)$ and A_n, in order to give an indication of the nature of the computation. In the final sections, we discuss the computa-

[1] If $\langle S\rangle = G$, then we write $\lambda_1(G, S)$ for λ_1 of this Cayley graph.

tions that were carried out. Section 4 describes the analysis of the generating pairs $\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\}$, viewed as elements of $SL_2(Z)$. Section 6 describes the distribution of second-largest eigenvalues over all 3-regular Cayley graphs for $PSL_2(F_{107})$ and A_{10}. The paper concludes by summarizing the results and suggesting directions for further investigation.

We would like to thank Alex Lubotzky for his suggestions and encouragement, as well as Larry Finkelstein for always answering our permutation group questions.

2 Cayley graphs and Fourier analysis

Let G be a finite group and let $S \subset G$ generate G. The *Cayley graph* $X = X(G, S)$ for G with respect to S is the undirected graph with vertex set equal to G, such that there is an edge between a and b in X if and only if $as = b$ for some $s \in S \bigcup S^{-1}$.

The *adjacency matrix* of X is the $|X|$ by $|X|$ matrix, with rows and columns indexed by vertices of X such that the x, y entry is 1 or 0 depending on whether or not x and y are adjacent. The spectrum of a graph is the spectrum of its adjacency matrix.

Assuming that X is k-regular, the largest eigenvalue of X is k. Let $\mu_1 \leq \cdots \leq \mu_{N-1} \leq k$ denote the eigenvalues of X. The *second-largest eigenvalue* of X, denoted $\lambda_1(X)$, is defined to be

$$\lambda_1 = \max_{\{i:\, |\mu_i| \neq k\}} |\mu_i| \,.$$

Various connectivity properties of a graph can be judged by studying its spectrum (see *e.g.*, [1]). Bounds for the diameter, expansion coefficient, and chromatic number all can be given in terms of λ_1. For details and references we refer the reader to other papers in this volume as well as the books ([9], [11]).

Fourier analysis provides an extremely useful tool for the study of the spectra of Cayley graphs. Recall that if f is any complex-valued function defined on G and ρ is any matrix representation of G, then the Fourier transform of f at ρ is defined to be the matrix sum

$$\widehat{f}(\rho) = \sum_{t \in G} f(t)\rho(t).$$

Let δ_S denote the characteristic function of the set $S \bigcup S^{-1}$. Then it is immediate that the adjacency matrix of $X(G, S)$ is equal to $\widehat{\delta_S}(\rho_{\mathrm{reg}})$, up to a reordering of the group elements, where ρ_{reg} denotes the right regular representation of G. Block diagonalization of the regular representation may be obtained using representation theory. If $\{\rho_1, \ldots, \rho_h\}$ is a complete set of inequivalent irreducible matrix representations of G, then there exists a change of basis such that

$$\widehat{\delta_S}(\rho_{\mathrm{reg}}) \sim \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_h \end{pmatrix}$$

where

$$B_i = \begin{pmatrix} \widehat{\delta_S}(\rho_i) & 0 & \cdots & 0 \\ 0 & \widehat{\delta_S}(\rho_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{\delta_S}(\rho_i) \end{pmatrix}$$

with $\deg(\rho_i)$ copies of $\widehat{\delta_S}(\rho_i)$ on the diagonal. Consequently,

$$\mathrm{spectrum}(X(G,S)) = \bigcup_{i=1}^{h} \mathrm{spectrum}(\widehat{\delta_S}(\rho_i)). \tag{1}$$

For numerical analysis of the spectrum, (2.1) provides a great advantage. Standard techniques for computing eigenvalues of a square matrix of degree d require $O(d^3)$ operations [13]. As the degree of the largest irreducible representation is bounded by $|\, G\,|^{1/2}$, (2.1) brings the computation from $O(|\, G\,|^3)$ operations to a more manageable $O(|G|\cdot(\max_i \deg(\rho_i)))$ operations.

In the case in which $S \bigcup S^{-1}$ is a union of conjugacy of classes of G, $\widehat{\delta_S}$ can be completely diagonalized and the diagonal elements computed as certain character sums [3]. However, if $S \bigcup S^{-1}$ is not a union of conjugacy classes (which is the situation in the computations considered here), then the actual matrix representations are required to compute the spectrum.

3 Representation theory for $SL_2(F_p)$, $PSL_2(F_p)$ and A_n.

In this section we give a very quick introduction and explanation of the representation theory of $SL_2(F_p)$, $PSL_2(F_p)$ and A_n. In-depth treatments can be found in [4] for A_n and [10] for $SL_2(F_p)$ and $PSL_2(F_p)$.

3.1 Representation theory for the alternating group

The representation theory of the alternating groups is most easily explained by using the representation theory of the symmetric groups.

A *partition* $\lambda \vdash n$ of a natural number n is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ with $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum_i \lambda_i = n$. The irreducible representations of the symmetric group S_n are parametrized by the partitions of n. For any such λ there is a corresponding *Ferrer's Diagram*, which is a left-justified arrangement of rows of boxes with λ_i boxes in row i. Figure 1 shows the Ferrer's diagram for the partition $(4, 2, 2, 1) \vdash 9$.

FIGURE 1: The Ferrer's diagram for $(4, 2, 2, 1)$

If $\lambda \vdash n$, then a *Young tableau* of shape λ is a Ferrer's diagram for λ with the numbers $1, \ldots, n$ inserted in the boxes. Two tableaux are said to be *row-equivalent* if any two corresponding rows contain the same sets of entries. A *tabloid* is an equivalence class of tableaux. S_n acts naturally on the set of all tabloids of a given shape, permuting the entries. The associated permutation module of S_n is denoted by M^λ. M^λ is reducible, except for the trivial partition (n). There is a natural way in which to pick out a specific irreducible component of M^λ, called the *Specht module*, and denoted S^λ. It is the action of S_n on S^λ which gives the irreducible representation ρ_λ associated to λ. For details see [4].

By restriction, any representation of S_n, gives a representation of A_n. It turns out that most of the ρ_λ remain irreducible upon restriction.

If $\lambda \vdash n$, then the *dual* partition, denoted λ' is the partition associated to the tranpose of the diagram for λ. For example, $(4, 2, 2, 1)' = (4, 3, 1, 1)$.

Theorem 3.1 *Let $\lambda, \nu \vdash n$. Then $\rho_\lambda \downarrow A_n = \rho_\nu \downarrow A_n$ iff either $\lambda = \nu$ or $\lambda' = \nu$. If $\lambda' \neq \lambda$ then $\rho_\lambda \downarrow A_n$ remains irreducible. Otherwise, $\rho_\lambda \downarrow A_n$ splits into two inequivalent irreducible components. This gives all the irreducible representations of A_n.*

Construction of the irreducible representations is straightforward, and orthogonal matrices (Young's orthogonal form) for the pairwise adjacent transpositions can be written down in terms of certain parameters from the tabloids ([4], Section 3.2). In fact for these elements, the matrices are even sparse, greatly expediting computations.

3.2 Representation theory for $SL_2(F_p)$ and $PSL_2(F_p)$

The irreducible representations of $SL_2(F_p)$ occur in two families, the *discrete series* and *principal series*. The distinction depends upon the restriction of an irreducible representation to the Borel subgroup $B < SL_2(F_p)$ of upper triangular matrices. An irreducible representation of $SL_2(F_p)$ is said to be from the principal series if its restriction to B contains the trivial representation. Otherwise, it is said to be from the discrete series.

Knowledge of the representations of $SL_2(F_p)$ gives the irreducible representations of $PSL_2(F_p)$.

Theorem 3.2 *Let ρ be an irreducible matrix representation of $SL_2(F_p)$ and let $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(F_p)$. If $-I$ is in the kernel of ρ, then ρ is constant on cosets $SL_2(F_p)/\{\pm I\}$ and as such gives an irreducible representation of $PSL_2(F_p)$. Under this identification, the set $\{\rho \mid \{\pm I\} \subset ker(\rho)\}$ gives a complete set of inequivalent irreducible representations of $PSL_2(F_p)$.*

As for the symmetric group, the irreducible representations for $SL_2(F_p)$ may be effciently constructed. Explicit contructions are given in [8], using the discussions in [10] and [12]. In short, all representations of the principal series can be constructed as induced 1-dimensional representations from B and as such are essentially in 1-1 correspondence with the characters of the diagonal subgroup, or split torus. The discrete series is less easily explained, but suffice it to say here that the representations are in close correspondence to the characters of the non-split torus in $SL_2(F_p)$ (cf. [10], Ch. 2, Section 5). From a computational point of view, the discrete series representations are much more expensive to construct. However, our experience has shown that the spectral properties of the principal and discrete series representations are almost identical. Consequently, in the computations discussed in Section 5 for computing all 3-regular Cayley graphs for $PSL_2(F_p)$, we actually carried out the computation by evaluating the Fourier transforms only at principal series representations.

4 Analysis of two infinite families of Cayley graphs

In this section we consider the spectrum for the sets of generators (of $SL_2(F_p)$)

$$\mathcal{G}_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}, \text{ and } \mathcal{G}_3 = \left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\}.$$

This extends our earlier work ([8], Section 5.2) and interestingly, shows that the open questions and speculations presented there also obtain for families which do not seem to admit analysis by Selberg's Theorem; that is, families which do not come from subgroups of finite index in $SL_2(Z)$.

Figure 2 plots $\lambda_1(\mathcal{G}_i, SL_2(F_p))$ versus p for $i = 2, 3$. The solid line at $\sqrt{3}/2$ marks the Ramanujan bound for these 4-regular graphs.

Figure 3 displays the full spectrum of $X(\mathcal{G}_i, SL_2(F_p))$ $(i = 2, 3)$ as a function of p. The graphs on the right display all eigenvalues occurring in discrete series representations of $SL_2(F_p)$ (with $\mathcal{G}_2$ on top and $\mathcal{G}_3$ on the bottom), while the graphs on the left show all which occur in principal series representations. That is, for every prime p, a single "dot" is displayed at the point (λ, p) if λ occurs as an eigenvalue for $X(\mathcal{G}_i, SL_2(F_p))$ in either principal series Fourier transform (graphs on the left of Figure 3) or a discrete series Fourier transform (graphs on the right of Figure 3). Note the remarkable similarity in the graphs. Even more than this appears to be true, as computation shows that if only the eigenvalues in a single Fourier transform are considered, then they are in fact generic, in the sense that if displayed as in Figure 3, they would yield similar pictures. See [8], Section 6, Question 2. Computational considerations for generating these graphs may be found in [8], Sections 3 and 4.

5 Distribution of λ_1 for A_{10} and $PSL_2(F_{107})$

In this section we compare the distribution of λ_1 for all 3-regular Cayley graphs for the group A_{10} with that of all 3-regular Cayley graphs for the comparably sized $PSL_2(F_{107})$. The motivation is to quantify the extent to which random Cayley graphs for the groups $PSL_2(F_p)$ and A_n are good expanders. While it is known by elementary combinatorial arguments that general random graphs are expanders, little is known about the behavior for Cayley graphs of the classical finite groups. The computation described in this section naturally extends the investigations of [8], where it was seen that 4-regular graphs for $SL_2(F_p)$ tend to have expansion coefficient close to the Ramanujan bound for p larger than 100. In this section we present the complete distribution of λ_1 for 3-regular Cayley graphs for $PSL_2(F_{107})$ and compare it to the corresponding distribution for A_{10}.

5.1 Determining generating pairs

For our analysis of all 3-regular Cayley graphs (cf. Section 3.2) we need an efficient test of whether or not, for a given $\{a, b\} \in G$, we have $\langle a, b\rangle = G$. In case $G = PSL_2(F_p)$ we use the classification of subgroups of $SL_2(F_p)$. In brief, this is done by checking the orbit structure of the action of the generators (lifted to $SL_2(F_p)$) on the projective line, where the action should be transitive. This is explained fully in [8].

For $G = A_n$, we use the fact [5] that if a permutation group on $n > 5$ points is transitive on 4-sets (4-homogeneous), then outside of a few exceptions, the group is either A_n or S_n. The orbit structure of the action of the group generated by two elements on the collection of 4-sets is easily determined on the computer.

To simplify the computation we take advantage of the fact that the spectrum is invariant under conjugation. If $S \subset G$, and $t \in G$ then let ${}^tS = tSt^{-1}$. Notice that $\lambda_1(G,S) = \lambda_1(G, {}^tS)$, since

$$\sum_{s' \in {}^tS} \rho_{\mathrm{reg}}(s') = \rho_{\mathrm{reg}}(t) \left(\sum_{s \in S} \rho_{\mathrm{reg}}(s) \right) \rho_{\mathrm{reg}}(t^{-1}).$$

Consequently, if $\alpha_1, \ldots, \alpha_k$ denote a set of conjugacy class representatives for involutions in G then at the very worst we need compute for all pairs of the form $\{\alpha_i, b\}$ for $b \in G$. A further reduction is obtained by then using the fact that for any $t \in Z_G(\alpha_i)$ (the centralizer of α_i), ${}^t\{\alpha_i, b\} = \{\alpha_i, {}^tb\}$. To implement these observations, for each involution α_i we decompose G into equivalence classes of orbits of $Z_G(\alpha_i)$. This can be accomplished in $O(|G|)$ operations. Then, we chose a representative b of each each orbit and compute the spectrum for the pair $\{\alpha_i, b\}$, computing a total of $|G| / |Z_G(\alpha_i)|$ eigenvalues.

5.2 The cumulative distribution of λ_1

Figure 4 presents the graphs of the cumulative distribution functions $F_G(\lambda)$ for $G = A_{10}$ and $G = PSL_2(F_{107})$ where

$$\begin{aligned} F_G(\lambda) = {} & \text{the fraction of generating pairs } S = \{a, b\} \text{ such that} \\ & \langle a, b \rangle = G \text{ and } a^2 = 1 \text{ with } \lambda_1(G, \{a, b\}) \leq \lambda. \end{aligned}$$

In our current implementation, working with representations of the discrete series is more computationally intensive by a factor of p than working with those of the principal series. Thus, the eigenvalues were computed only from the principal series. This approximation was justified by our experience [8] that the principal and discrete series representations are essentially identical from the point of view of spectral analysis of Cayley graphs.

The figure shows that the eigenvalues for $PSL_2(F_{107})$ are more sharply concentrated near the Ramanujan bound of $\frac{2\sqrt{2}}{3} \approx 0.942809$ than those for A_{10}. As Figure 8 of [8] indicates, this concentration becomes much greater for $PSL_2(F_p)$ as p grows larger. The figure suggests that the distribution of second-largest eigenvalues is qualitatively different for the two groups. However, the computation is, of course, inconclusive, and further investigation of the asymptotics will be required to resolve the question.

As discussed above, the computation was carried out by choosing for each involution, a representative of each orbit of the associated centralizer. Since $PSL_2(F_{107})$ has only one conjugacy class of involutions, while A_{10} has two, it is conceivable that the two classes in A_{10} may behave differently with respect to expansion. We examined this possibility by comparing the respective spectra. Figure 5 plots λ_1 for each of these conjugacy classes. It is seen that the distribution of eigenvalues associated with the involution (12)(34) is characteristic of the combined distribution. This is to be expected, however, due to the fact that the number of conjugacy classes of generating pairs is 1496 for this involution, while there are only 82 such classes for the involution (12)(34)(56)(78).

Finally, it has been observed that the action of $SL_2(F_p)$ on the projective line approximates the action of $SL_2(F_p)$ on itself ([2], [8]). That is, $SL_2(F_p)$ acts naturally on $P^1(F_p)$. Let $\langle a, b\rangle = SL_2(F_p)$ and consider the graph with vertex set $P^1(F_p)$ and $v \sim w$ iff $sv = w$ for some $s \in \{a, b\} \cup \{a, b\}^{-1}$. The conjecture is that λ_1 for this graph closely approximates $\lambda_1(G, \{a, b\})$. This is equivalent to saying that the largest nontrivial eigenvalue occurring in the Fourier transform at $1 \uparrow_B^{\mathrm{SL}_2(F_p)}$ is close to $\lambda_1(G, \{a, b\})$. The analogous case for symmetric groups would be that natural permutation representation of A_n acting on $\{1, \ldots, n\}$ approximates the action of the full set of irreducible representations. Figure 6 seems to show that this is not the case for A_{10}.

6 Conclusions and open questions

The computations that have been described in this paper cast a faint light on the spectral geometry of finite groups. Clearly, further theorems are needed to illuminate the phenomena that we see in the data. In this section we suggest several areas for further investigation.

Our computations suggest that there is a significant difference in the asymptotic behavior of the spectra of Cayley graphs for $PSL_2(F_p)$ and A_n. This difference should be quantified. For example, it is natural to ask whether a lower bound of the form

$$\liminf_{n,p\to\infty} \| F_\lambda(PSL_2(F_p)) - F_\lambda(A_n) \|_1 > c$$

can be obtained, where the L^1 distance could be replace by any other suitable measure of similarity, such as the Kullback-Liebler distance.

In this paper we have restricted our attention to 3-regular Cayley graphs, where one of the generating elements is an involution. It is natural to ask if a general k-regular Cayley graph enjoys the same properties that appear to hold in the cases we have considered. In particular, it would be of interest to determine whether or not a k-regular Cayley graph for $PSL_2(F_p)$ is a good expander with high probability as $p \to \infty$.

In addition, we have observed [8] that the action of $PSL_2(F_p)$ on itself is closely approximated by its action on the projective line $P^1(F_p)$, but an analogous statement is perhaps not true for A_n. Thus, the question arises: for which classical finite groups G is there a finite set S on which G acts, with $\frac{|S|}{|G|} = o(1)$ as $|G| \to \infty$, such that the spectrum of $X(G, \{a, b\})$ is closely approximated by the graph of the action of $\{a, b\}$ on S? Since the computational payoff from having such an approximation is so great, an understanding of this question would be of considerable interest.

Finally, our computations show that the generating sets $\mathcal{G}_2$ and $\mathcal{G}_3$ for $SL_2(F_p)$ result in Cayley graphs having identical spectral properties, in spite of the fact that Selberg's theorem only applies to the first set. Thus, we are led to inquire to what extent the property of being a quotient of a subgroup of finite index in $SL_2(Z)$ determines expansion properties. This example would also seem to indicate that there should be a proof of the expansion properties for such families that does not depend on Selberg's theorem.

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FIGURE 2: $\lambda_1(\mathcal{G}_2, SL_2(F_p))$ (left), $\lambda_1(\mathcal{G}_3, SL_2(F_p))$ (right)

FIGURE 4: $F_\lambda(PSL_2(F_{107}))$ (dashed); $F_\lambda(A_{10})$ (solid)

FIGURE 5: λ_1 for 3-regular Cayley graphs on A_{10},
dashed–involution (12)(34); solid–involution (12)(34)(56)(78)

FIGURE 6: λ_1 for the action on the permutation graph for A_{10},
dashed–involution (12)(34); solid–involution (12)(34)(56)(78)