Two's Company, Three's a Crowd: Stable Family and Threesome Roommates Problems*

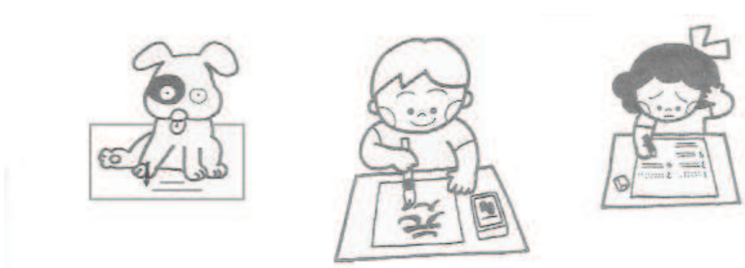
Chien-Chung Huang

Dartmouth College
villars@cs.dartmouth.edu

Abstract. We investigate Knuth's eleventh open question on stable matchings. In the stable family problem, sets of women, men, and dogs are given, all of whom state their preferences among the other two groups. The goal is to organize them into family units, so that no three of them have the incentive to desert their assigned family members to form a new family. A similar problem, called the threesome roommates problem, assumes that a group of persons, each with their preferences among the combinations of two others, are to be partitioned into triples. Similarly, the goal is to make sure that no three persons want to break up with their assigned roommates.

Ng and Hirschberg were the first to investigate these two problems. In their formulation, each participant provides a strictly-ordered list of all combinations. They proved that under this scheme, both problems are NP-complete. Their paper reviewers pointed out that their reduction exploits *inconsistent* preference lists and they wonder whether these two problems remain NP-complete if preferences are required to be consistent. We answer in the affirmative.

In order to give these two problems a broader outlook, we also consider the possibility that participants can express indifference, on the condition that the preference consistency has to be maintained. As an example, we propose a scheme in which all participants submit two (or just one in the roommates case) lists ranking the other two groups separately. The order of the combinations is decided by the sum of their ordinal numbers. Combinations are tied when the sums are equal. By introducing indifference, a hierarchy of stabilities can be defined. We prove that all stability definitions lead to NP-completeness for existence of a stable matching.



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1 Problem Definition

Knuth proposed twelve open questions on the stable matching problem [9]. The eleventh question asks whether the well-studied stable marriage problem [4] can be generalized to the case of three parties, women, men, and dogs. In this paper, we call this problem the *stable family* problem and refer generically to all participants in this problem as "players." Roughly speaking, given sets of women, men, and dogs, all of whom state their preferences among the other two groups, the goal is to organize them into family units so that there is no *blocking triple*: three players each preferring one another to their assigned family members. A problem in a similar vein, which we call the *threesome roommates* problem, assumes that $3n$ students are to be assigned to the dormitory bedrooms in some college. They state their preferences of the combinations of two other persons. The goal is to partition them into sets of size 3. Such a partition (matching) is said to be stable if no three persons each prefer the others to their assigned roommates.

As Knuth does not specify any precise definition of "preference" and "blocking triples," one can conceive a number of ways to define the two problems. One possible formulation is that each player submits a strictly-ordered preference list, ranking all possible combinations that she/he/it can get in a matching. We call such a scheme strictly-ordered-complete-list (**SOCL**) scheme. In this setting, Ng and Hirschberg [10] proved that both problems are NP-complete.

At the end of their paper, Ng and Hirschberg mentioned that their reviewers pointed out their reduction allows preference to be *inconsistent*. For example, man m might rank (w_1, d_1) higher than (w_2, d_1), but he also ranks (w_2, d_2) higher than (w_1, d_2). In other words, he does not consistently prefer woman w_1 over woman w_2 (nor the other way around). Independently, Subramanian [11] gave an alternative NP-completeness proof for stable family, but his reduction also uses inconsistent lists.

The reviewers of Ng and Hirschberg wondered whether these two problems remain NP-complete if inconsistency is disallowed. To answer this open question and to motivate some variants problems we will define, we introduce the notion of *preference posets* and *simple lists*. In stable family, assuming that each player has two simple lists in which two different types of players are ranked separately, a preference poset is a product poset of the two simple lists. In such a poset, the combination (w_1, d_1) precedes another combination (w_2, d_2) only if w_1 ranks at least as high as w_2 and d_1 at least as high as d_2 in the simple lists. If neither combination precedes the other, they are incomparable. Similarly, in threesome roommates, the preference poset is the product poset of the one simple list with itself. By this notion, the question raised by the reviewers of Ng and Hirschberg can be rephrased as follows. Under the **SOCL** scheme, if every player has to submit a preference list which is a *linear extension* of her/his/its preference poset, are the stable family and the threesome roommates still NP-complete? We answer in the affirmative.

In an attempt to give these two problems a broader outlook, we then allow players to express indifference by giving full preference lists containing ties. In particular, to capture the spirit of maintaining consistency in the preferences, we stipulate that the full list must be a *relaxed linear extension* of a preference poset: strict precedence order in the poset has to be observed in the relaxed linear extension; only incomparable elements in the poset can be tied.

We propose the following scheme to make the above concept concrete. Suppose that a player submits two simple lists (or just one in the roommates case). We create a full list, ranking the combinations based on the sums of their ordinal numbers. For example, for man m, the combination of his rank-2 woman and rank-5 dog is as good as that of his rank-4 woman and rank-3 dog; while both of them are inferior the combination of his top-ranked woman and his top-ranked dog. We call such a scheme precedence-by-ordinal-number (**PON**) scheme. The **PON** scheme produces full preference lists which are relaxed linear extensions of preference posets. Also, one can envisage an even more flexible scheme. For example, instead of giving "ranks," the players can provide "ratings" of other players. The order of the combinations can be decided by the sum of the ratings; two combinations are tied only when the sums of their ratings are equivalent. Setting theoretical concerns aside for a moment, the above schemes are probably more practicable when n is large, because a player only has to provide lists of $\Theta(n)$ length, while under the **SOCL** scheme, they have to give strictly ordered lists of size $\Theta(n^2)$.

By allowing indifference, we can define 4 different types of blocking triples and, based on them, build up a hierarchy of stabilities. (This hierarchy is similar to that constructed by Irving in the context of 2-party stable matchings [7].)

– Weak Stable Matching: a blocking triple is one in which all three players of the blocking triple strictly prefer the other two members in the triple over their assigned family members (roommates).
– Strong Stable Matching: a blocking triple is one in which at least two players of the blocking triple strictly prefer the other two players in the triple to their assigned family members (roommates), while the remaining player can be indifferent or also strictly prefer the other two players in the triple.
– Super Stable Matching: a blocking triple is one in which at least one player of the blocking triple strictly prefers the other two players in the triple to her/his/its assigned family members (roommates), while the remaining players can be indifferent or also strictly prefer the other two players in the triple.
– Ultra Stable Matching: a blocking triple is one in which all three players in the triple are at least indifferent to the others.

Note that if ties are not allowed in the full preference lists, i.e., the **SOCL** scheme, then blocking triples can only be of degree 3. Thus there can be only one type of stability. For presentational reason, in this case, we refer to the stability under the **SOCL** scheme as the weak stability.

Our Results and Paper Roadmap We will prove in the paper that, if full preference lists are (relaxed) linear extensions of preference posets, the problem of deciding whether weak/strong/super/ultra stable matchings exist is NP-complete in both the stable family problem and the threesome roommates problem. Our reduction techniques are inspired by Ng and Hirschberg's, although the consistency requirement in the preferences makes our construction more involved. In presenting our result, instead of directly answering the open question posed by Ng and Hirschberg's reviewers by studying weak-stability, we make a detour to first study strong/super/ultra stability. Introducing them first helps us to explain our intuition behind the more complex reduction for the former problem.

As is well-known, the stable marriage and the stable roommates problems can be solved in $O(n^2)$ time, by the Gale-Shapley algorithm [4] and by the Irving algorithm [6], respectively. Unfortunately, our results, along with Ng and Hirschberg and Subramanian's, indicate that attempts to efficiently solve the stable matching problem in generalized cases of three (or more) parties are unlikely to be fruitful. This is not surprising, as in theoretical computer science, the fine line between **P** and **NP** is often drawn between the numbers two and three.

We organize the paper as follows. In Section 2, we present necessary notation and some basic lemmas on the properties of posets; Section 3 proves the NP-completeness of strong/super/ultra stable matchings in the stable family problem under the **PON** scheme; Section 4 presents a reduction to transform a stable family problem to a threesome roommate problem, thereby establishing the NP-completeness of strong/super/ultra stable matchings in the latter; Section 5 considers the **SOCL** scheme in threesome roommates and exhibits another reduction to show the NP-completeness of (weak) stable matching, thereby answering the open question posed by the anonymous reviewers of Ng and Hirschberg. Section 6 concludes and discusses related issues.

2 Preliminaries

We use $\mathcal{M}$, $\mathcal{W}$, $\mathcal{D}$ to indicate the sets of men, women, and dogs in stable family; the students in threesome roommates are denoted as $\mathcal{R}$. In stable family, $L_g(p)$ denotes the simple list of player p on the players of type $g \in \{\mathcal{M}, \mathcal{W}, \mathcal{D}\}$. For example $L_{\mathcal{W}}(m)$ is the simple list of man m among women $\mathcal{W}$. In threesome roommates, we simply write $L(m)$, where $m \in \mathcal{R}$, dropping the subscript.

In general, we use the notation $\succ$ to denote the precedence order (in either posets or in linear lists). For example, supposing that p_i ranks higher than p_j in the list l, we write $p_i \succ_l p_j$. In a poset Q, two elements q_i, q_j either one precedes the other, which we write $q_i \succ_Q q_j$ or $q_j \succ_Q q_i$, or they are incomparable, which is expressed as $q_i||_Q q_j$. The notation $\succ$ is also used to express explicitly the order of players in the simple lists. For example, we write $L(p) = q \succ r \succ \cdots$ to show that player p prefers player q to player r. Note also that the notation $\cdots$ denotes the remaining players in arbitrary order.

We say a blocking triple is of degree i, if i players strictly prefer the triple while the remaining $3-i$ players are indifferent. Unless stated otherwise, in the article, when we say some triple "blocks," it is always a blocking triple of degree 3. We use the notation $r_p(q)$ to indicate the rank of q on player p's simple list.

A preference poset constructed from lists l_1 and l_2 is written as $l_1 \times l_2$. To be precise, given lists l_1 and l_2 and the poset $l_1 \times l_2$, supposing that $\{p_i, p_j\}, \{p_{i'}, p_{j'}\} \in l_1 \times l_2$, then $\{p_i, p_j\} \succ_{l_1 \times l_2} \{p_{i'}, p_{j'}\}$ only if (1) $p_i \succ_{l_1} p_{i'}, p_j = p_{j'}$, or (2) $p_j \succ_{l_2} p_{j'}, p_i = p_{i'}$, or (3) $p_i \succ_{l_1} p_{i'}, p_j \succ_{l_2} p_{j'}$. The notation $\pi(X)$ means an arbitrary permutation of elements in the set X. $E_\pi(l_1 \times l_2)$ is an arbitrary linear extension of the preference poset $l_1 \times l_2$.

In a poset, we call an element a *pivot* if in the linear extension of the poset we will create, this element precedes all its incomparable elements. Any element can be a pivot, as will be shown by the lemma below.

Lemma 1. *Given any poset Q and any element $q \in Q$, there exists a linear extension l of Q such that if $q||_Q q'$, then $q \succ_l q'$.*

The next lemma will be useful when we present the reduction for the threesome roommates problem.

Lemma 2. *Let l be a strictly-ordered list. Suppose that l is decomposed into nonempty contiguous sublists $(l_1, l_2, \cdots, l_k)$ such that (1) $\bigcup_{i=1}^{k} l_i = l$, (2) if $e \succ_{l_i} f$, then $e \succ_l f$, and (3) if $e \in l_i, f \in l_j, i < j$, then $e \succ_l f$. Then there exists a linear extension of $l \times l$ such that all combinations drawn from $\{l_i, l_j\}$ precede all pairs drawn from $\{l_{i'}, l_{j'}\}$, provided that $i \leq j, i' \leq j'$ and one of the following conditions holds (1) $i < i'$, (2) $i = i', j < j'$.*

The proofs of the two lemmas can be found in the appendix.

3 Reducing Three-dimensional Matching to Stable Family

In this section, we focus on the NP-completeness of strong stable matching under the **PON** scheme. Similar results hold for super stable and ultra stable matchings by a straightforward argument and will be discussed at the end of this section.

Our reduction is from the three-dimensional matching problem, one of the 21 NP-complete problems in Karp's seminal paper [8]. The problem instance is given in the form $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \mathcal{T})$, where $\mathcal{T} \subseteq \mathcal{M} \times \mathcal{W} \times \mathcal{D}$. The goal is to decide whether a perfect matching $\mathcal{M} \subseteq \mathcal{T}$ exists. This problem remains NP-complete even if every player in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ appears exactly 2 or 3 times in the triples of $\mathcal{T}$ [5].

We first explain the intuition behind our reduction. Supposing that man m_i appears in three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$, we create three *dopplegangers*, m_{i1}, m_{i2}, m_{i3} in the derived stable family problem instance Υ'. We also create four garbage collectors, $w^g_{i1}, d^g_{i1}, w^g_{i2}, d^g_{i2}$. Each doppleganger m_{ij} puts a woman-dog pair, with whom man m_i shares a triple, and the garbage collectors on top of his two simple lists. The goal of our design is that in a stable matching, exactly one doppleganger will be matched to a woman-dog pair with whom m_i shares a triple in $\mathcal{T}$, while the other two dopplegangers will be matched to garbage collectors. In the case that there are only two triples in $\mathcal{T}$ containing man m_i, we artificially make a copy of one of the triples, making the total number of triples three, and treat him as described above.

Now, we will refer to the set of dopplegangers as $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, the set of garbage collectors as $\mathcal{W}^g_1, \mathcal{W}^g_2, \mathcal{D}^g_1, \mathcal{D}^g_2$ and the original set of real women and real dogs as $\mathcal{W}, \mathcal{D}$. Collectively, we refer to them as $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}^g_1 \cup \mathcal{W}^g_2 \cup \mathcal{W} \cup \mathcal{D}^g_1 \cup \mathcal{D}^g_2 \cup \mathcal{D}$.

To realize our plan, we introduce two gadgets. The first is three sets of "dummy players": $m^{\#}_1, w^{\#}_1, d^{\#}_1, m^{\#}_2, w^{\#}_2, d^{\#}_2, m^{\#}_3, w^{\#}_3, d^{\#}_3$. Their preferences are such that they must be matched to one another in a stable matching. To be precise, for $j \in \{1, 2, 3\}$,

– $L_{\mathcal{W}}(m_j^{\#}) = w_j^{\#} \succ \cdots, L_{\mathcal{D}}(m_j^{\#}) = d_j^{\#} \succ \cdots$
– $L_{\mathcal{M}}(w_j^{\#}) = m_j^{\#} \succ \cdots, L_{\mathcal{D}}(w_j^{\#}) = d_j^{\#} \succ \cdots$
– $L_{\mathcal{M}}(d_j^{\#}) = m_j^{\#} \succ \cdots, L_{\mathcal{W}}(d_j^{\#}) = w_j^{\#} \succ \cdots$

These nine dummy players are used to "pad" the preference lists of other players. Their purpose will be clear shortly.

Another gadget we need is a set of "guard players" for each doppleganger in $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. They will make sure that in a stable matching, a doppleganger m_{ij} will only get a woman-dog pair with whom m_i shares a triple in $\mathcal{T}$ or those garbage collectors. As an example, consider the doppleganger m_{i1}. He has six associated guard players, $m_{i1}^{\flat 1}, w_{i1}^{\flat 1}, d_{i1}^{\flat 1}, m_{i1}^{\flat 2}, w_{i1}^{\flat 2}, d_{i1}^{\flat 2}$ and their preferences are summarized below:

– $L_{\mathcal{W}}(m_{i1}) = w_{i2}^g \succ w_{i1}^g \succ w_{ia} \succ w_{i1}^{\flat 1} \succ w_{i1}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots,$
  $L_{\mathcal{D}}(m_{i1}) = d_{i2}^g \succ d_{i1}^g \succ d_{ia} \succ d_{i1}^{\flat 2} \succ d_{i1}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
– $L_{\mathcal{W}}(m_{i1}^{\flat 1}) = w_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{D}}(m_{i1}^{\flat 1}) = d_{i1}^{\flat 1} \succ \cdots$
  $L_{\mathcal{W}}(m_{i1}^{\flat 2}) = w_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{D}}(m_{i1}^{\flat 2}) = d_{i1}^{\flat 2} \succ \cdots$
– $L_{\mathcal{M}}(w_{i1}^{\flat 1}) = m_{i1} \succ m_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{D}}(w_{i1}^{\flat 1}) = d_{i1}^{\flat 1} \succ d_1^{\#} \succ \cdots$
  $L_{\mathcal{M}}(w_{i1}^{\flat 2}) = m_{i1} \succ m_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{D}}(w_{i1}^{\flat 2}) = d_{i1}^{\flat 2} \succ d_1^{\#} \succ \cdots$
– $L_{\mathcal{M}}(d_{i1}^{\flat 1}) = m_{i1} \succ m_{i1}^{\flat 1} \succ \cdots, L_{\mathcal{W}}(d_{i1}^{\flat 1}) = w_{i1}^{\flat 1} \succ w_1^{\#} \succ \cdots$
  $L_{\mathcal{M}}(d_{i1}^{\flat 2}) = m_{i1} \succ m_{i1}^{\flat 2} \succ \cdots, L_{\mathcal{W}}(d_{i1}^{\flat 2}) = w_{i1}^{\flat 2} \succ w_1^{\#} \succ \cdots$

The following case analysis proves that, in a stable matching M', m_{i1} will get only players from the set $\{w_{i1}^g, w_{i2}^g, w_{ia}, d_{i1}^g, d_{i2}^g, d_{ia}\}$.

– Suppose that m_{i1} gets two players ranking below $w_{i1}^{\flat 1}$ and $d_{i1}^{\flat 1}$ respectively. It can be observed that for both $w_{i1}^{\flat 1}, d_{i1}^{\flat 1}$, the best man is m_{i1}. Therefore, they would prefer m_{i1} and so does he them, inducing a blocking triple to M', a contradiction.
– Suppose that m_{i1} gets a woman $w \in \{w_{ia}, w_{i1}^g, w_{i2}^g\}$ and a dog d ranking below $d_{i1}^{\flat 1}$. In this case, we can be sure that d cannot be $d_1^{\#}$ or $d_2^{\#}$ or $d_3^{\#}$, since their preferences guarantee that they will only be matched to other dummy players. So, $r_{m_{i1}}(w) + r_{m_{i1}}(d) \geq 10$, while $r_{m_{i1}}(w_{i1}^{\flat 1}) + r_{m_{i1}}(d_{i1}^{\flat 1}) = 9$, causing $(m_{i1}, w_{i1}^{\flat 1}, d_{i1}^{\flat 1})$ to become a blocking triple. This example explains why we need to "pad" the simple lists of m_{i1} with dummy players.
  The case that m_{i1} gets a dog $d \in \{d_{ia}, d_{i1}^g, d_{i2}^g\}$ and a woman w ranking lower than $w_{i1}^{\flat 2}$ follows analogus arguments; $(m_{i1}, w_{i2}^g, d_{i2}^g)$ will become a blocking triple.
– Suppose that m_{i1} gets only one of the players from the set $\{w_{i1}^{\flat 1}, w_{i1}^{\flat 2}, d_{i1}^{\flat 1}, d_{i1}^{\flat 2}\}$. Without loss of generality, we assume that $(m_{i1}, w_{i1}^{\flat 1}, d^{\phi})$, where $d^{\phi} \neq d_{i1}^{\flat 1}$, is part of the matching. For woman $w_{i1}^{\flat 1}$, dog d^{ϕ} cannot be the dummy player $d_1^{\#}$. Therefore, $r_{w_{i1}^{\flat 1}}(m_{i1}) + r_{w_{i1}^{\flat 1}}(d^{\phi}) \geq 4 > 3 = r_{w_{i1}^{\flat 1}}(m_{i1}^{\flat 1}) + r_{w_{i1}^{\flat 1}}(d_{i1}^{\flat 1})$. Similarly for $d_{i1}^{\flat 1}$, $r_{d_{i1}^{\flat 1}}(m_{i1}^{\flat 1}) + r_{d_{i1}^{\flat 1}}(w_{i1}^{\flat 1}) = 3$, which is better than whatever combination it can get. Therefore, we have that $(m_{i1}^{\flat 1}, w_{i1}^{\flat 1}, d_{i1}^{\flat 1})$ constitutes a blocking triple to M'. This example shows why we need to pad the preference of $w_{i1}^{\flat 1}, d_{i1}^{\flat 1}$ (and also $w_{i1}^{\flat 2}, d_{i1}^{\flat 2}$) with dummy players.

– Suppose that m_{i1} gets $w_{i1}^{\flat 1}$ and $d_{i1}^{\flat 1}$. Note that $w_{i1}^{\flat 1} \succ w_{i1}^{\flat 2}$ and $d_{i1}^{\flat 2} \succ d_{i1}^{\flat 1}$. Therefore, m_{i1} is *indifferent* to the combinations of $w_{i1}^{\flat 2}$ and $d_{i1}^{\flat 2}$, since $r_{m_{i1}}(w_{i1}^{\flat 1}) + r_{m_{i1}}(d_{i1}^{\flat 1}) = 9 = r_{m_{i1}}(w_{i1}^{\flat 2}) + r_{m_{i1}}(d_{i1}^{\flat 2})$. Additionally, $w_{i1}^{\flat 2}, d_{i1}^{\flat 2}$ strictly prefer m_{i1}. Hence $(m_{i1}, w_{i1}^{\flat 2}, d_{i1}^{\flat 2})$ constitutes a blocking triple of degree 2 to M'. This explains why we need two sets of guard players to guarantee that the doppleganger will "behave" in a stable matching.

  Again, the case that m_{i1} gets $w_{i1}^{\flat 2}$ and $d_{i1}^{\flat 2}$ follows analogus arguments.

The other two dopplegangers m_{i2}, m_{i3} also have six associated guard players for each; they, along with their associated guard players, have similar preferences to guarantee that m_{i2} and m_{i3} will only get garbage collectors or the woman-dog pairs with whom m_i shares triples. The only difference in the lists is that m_{i2} and m_{i3} replace w_{ia}, d_{ia} with w_{ib}, d_{ib}, and with w_{ic}, d_{ic}, respectively, in their simple lists. For a summary of the simple lists of members in the set X, see Table 1. It should be noted that w_{i1}^g, d_{i1}^g (and also w_{i2}^g, d_{i2}^g) rank the three dopplegangers in reverse order. This trick guarantees that the dopplegangers will not form blocking triples with the garbage collectors, defeating our purpose. For example, suppose (m_{i1}, w_{ia}, d_{ia}) is part of the matching, we want to avoid $(m_{i1}, w_{i1}^g, d_{i1}^g)$ to becoming a blocking triple. It can be easily verified that if w_{i1}^g and d_{i1}^g are matched to m_{i2} or m_{i3}, such a blocking triple will not be formed.

Table 1. The simple lists of all players in the set $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$. We assume that there exist three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$.

Player	Simple Lists
$m_{i1} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i1}) = w_{i2}^g \succ w_{i1}^g \succ w_{ia} \succ w_{i1}^{\flat 1} \succ w_{i1}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i1}) = d_{i2}^g \succ d_{i1}^g \succ d_{ia} \succ d_{i1}^{\flat 2} \succ d_{i1}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$m_{i2} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i2}) = w_{i2}^g \succ w_{i1}^g \succ w_{ib} \succ w_{i2}^{\flat 1} \succ w_{i2}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i2}) = d_{i2}^g \succ d_{i1}^g \succ d_{ib} \succ d_{i2}^{\flat 2} \succ d_{i2}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$m_{i3} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i3}) = w_{i2}^g \succ w_{i1}^g \succ w_{ic} \succ w_{i3}^{\flat 1} \succ w_{i3}^{\flat 2} \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$ $L_{\mathcal{D}}(m_{i3}) = d_{i2}^g \succ d_{i1}^g \succ d_{ic} \succ d_{i3}^{\flat 2} \succ d_{i3}^{\flat 1} \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$w_{i1}^g \in \mathcal{W}_1^g$	$L_{\mathcal{M}}(w_{i1}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ \cdots$ $L_{\mathcal{D}}(w_{i1}^g) = d_{i1}^g \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$d_{i1}^g \in \mathcal{D}_1^g$	$L_{\mathcal{M}}(d_{i1}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ \cdots$ $L_{\mathcal{W}}(d_{i1}^g) = w_{i1}^g \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$
$w_{i2}^g \in \mathcal{W}_2^g$	$L_{\mathcal{M}}(w_{i2}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ \cdots$ $L_{\mathcal{D}}(w_{i2}^g) = d_{i2}^g \succ d_1^{\#} \succ d_2^{\#} \succ d_3^{\#} \succ \cdots$
$d_{i2}^g \in \mathcal{D}_2^g$	$L_{\mathcal{M}}(d_{i2}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ \cdots$ $L_{\mathcal{W}}(d_{i2}^g) = w_{i2}^g \succ w_1^{\#} \succ w_2^{\#} \succ w_3^{\#} \succ \cdots$
$w \in \mathcal{W}$	$L_{\mathcal{M}}(w) = \cdots$ $L_{\mathcal{D}}(w) = \cdots$
$d \in \mathcal{D}$	$L_{\mathcal{M}}(d) = \cdots$ $L_{\mathcal{W}}(d) = \cdots$

Finally, garbage collectors also use dummy players to pad their simple lists, to avoid the awkward situation that some doppleganger is matched to a real woman and a garbage collector dog (or a real dog and a garbage collector woman). How this arrangement works will be clear in the proof below.

Lemma 3. *Suppose a stable matching M' exists in the derived stable family problem instance Υ'. The following facts hold in M':*

- *Fact A: The three sets of dummy players are matched to one another.*
- *Fact B: For each doppleganger $m_{ij} \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$, the ranks of his family members are at least as high as 3 in his simple lists.*
- *Fact C: The six associated guard players of each doppleganger $m_{ij} \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ are matched to one another.*

Proof. Fact A follows directly from construction. Fact B is true as we have argued in the case analysis before. Fact C is true because if the guard players are not matched to one another, they will block M', unless $w_{ij}^{\flat 1}, d_{ij}^{\flat 1}$ or $w_{ij}^{\flat 2}, d_{ij}^{\flat 2}$ are matched to m_{ij} in M', but this is impossible because of Fact B. □

Lemma 4. *Suppose a stable matching M' exists in the derived stable family problem instance Υ'. Consider the garbage collectors $w_{i1}^g, d_{i1}^g, w_{i2}^g, d_{i2}^g$ created for man $m_i \in \mathcal{M}$. We must have that w_{i1}^g, d_{i1}^g belong to the same triple t_1 and that w_{i2}^g, d_{i2}^g belong to the same triple t_2 in M'. Moreover, in t_1 and t_2, the man player must be one of the dopplegangers m_{i1}, m_{i2} and m_{i3}.*

Proof. We will prove this lemma by progressively establishing the following facts.

Fact D: *w_{i2}^g and d_{i2}^g must belong to the same triple t_2 in M.*

Proof: For a contradiction, suppose that w_{i2}^g and d_{i2}^g are in different triples in M'. We claim that $(m_{i1}, w_{i2}^g, d_{i2}^g)$ forms a blocking triple. It is obvious that m_{i1} and w_{i2}^g prefer such a triple. Now let the man and woman partners of d_{i2}^g be m^ϕ and $w^\phi \neq w_{i2}^g$; then by Fact A in Lemma 3, $r_{d_{i2}^g}(w) \geq 5$. We have that $r_{d_{i2}^g}(m_{i1}) + r_{d_{i2}^g}(w_{i2}^g) = 4 < 6 \leq r_{d_{i2}^g}(m^\phi) + r_{d_{i2}^g}(w^\phi)$. So d_{i2}^g will also prefer m_{i1} and w_{i2}^g, forming a blocking triple with them to M'. This proof also shows why we need to pad the preferences of the garbage collectors.

Fact E: *w_{i1}^g and d_{i1}^g must belong to the same triple t_1 in M'.*

Proof: For a contradiction, suppose that $(m^{\phi 1}, w_{i1}^g, d^{\phi 1})$ and $(m^{\phi 2}, w^{\phi 2}, d_{i1}^g)$ are triples in M'. There exists at least one doppleganger in $\{m_{i1}, m_{i2}, m_{i3}\}$ preferring the combination of w_{i1}^g and d_{i1}^g (since at most one doppleganger can be matched to w_{i2}^g and d_{i2}^g.) Let such a doppleganger be m_{ij}. Then by Fact A in Lemma 3, $r_{w_{i1}^g}(m_{ij}) + r_{w_{i1}^g}(d_{i1}^g) \leq 4 < 6 \leq r_{w_{i1}^g}(m^{\phi 1}) + r_{w_{i1}^g}(d^{\phi 2})$; and similarly, $r_{d_{i1}^g}(m_{ij}) + r_{d_{i1}^g}(w_{i1}^g) \leq 4 < 6 \leq r_{d_{i1}^g}(m^{\phi 2}) + r_{d_{i1}^g}(w^{\phi 2})$, implying that $(m_{ij}, w_{i1}^g, d_{i1}^g)$ blocks M.

***Fact F**: w_{i2}^g and d_{i2}^g must be matched to one of the dopplegangers in M, and so are w_{i1}^g and d_{i1}^g.*

Proof: If w_{i2}^g and d_{i2}^g are not matched to a doppleganger of m_i, then any doppleganger m_{ij} will prefer the combination of them over his family members, causing $(m_{ij}, w_{i2}^g, d_{i2}^g)$ to block M. A similar argument applies to the case of w_{i1}^g and d_{i1}^g, giving the lemma. □

By the previous two lemmas, we have established the correctness of the reduction on one side.

Lemma 5. *(Sufficiency) If there exists a stable matching M' in the derived stable family problem instance Υ', there exists a perfect matching M in the original three-dimensional matching instance Υ.*

To show the necessity, we need to prove one more lemma.

Lemma 6. *In a matching M' in the derived stable family problem instance Υ', suppose that dummy players are matched to one another. Suppose further that the garbage collectors of m_i are matched to two of the dopplegangers, while the remaining doppleganger m_{ij} is matched to a real woman and a real dog with whom m_i shares a triple in $\mathcal{T}$ in the original three-dimensional matching instance Υ. Then there is no blocking triple in which the dopplegangers m_{i1}, m_{i2}, and m_{i3} are involved.*

Proof. We assume that $(m_{i1}, w_{i2}^g, d_{i2}^g), (m_{i2}, w_{i1}^g, d_{i1}^g), (m_{i3}, w_{ic}, d_{ic}) \in M'$. Other cases follow analogous arguments. We claim that there does not exist a blocking triple of the form $(m_{ij}, w_{i1}^g, d^{\phi 1})$, $(m_{ij}, w_{i2}^g, d^{\phi 2})$, $(m_{ij}, w^{\phi 3}, d_{i1}^g)$, and $(m_{ij}, w^{\phi 4}, d_{i2}^g)$ where $d^{\phi 1} \neq d_{i1}^g$, $d^{\phi 2} \neq d_{i2}^g$, $w^{\phi 3} \neq w_{i1}^g$, and $w^{\phi 4} \neq w_{i2}^g$. We only argue the first case. Since $d^{\phi 1} \notin \{d_1^\#, d_2^\#, d_3^\#\}$, we have $r_{w_{i1}^g}(d^{\phi 1}) \geq 5 > 3 = r_{w_{i1}^g}(d_{i1}^g) + r_{w_{i1}^g}(m_{i2})$. Therefore, w_{i1}^g has no incentive to join the combination of m_{ij} and $d^{\phi 1}$. So we only need to consider the three following potential blocking triples: $(m_{i2}, w_{i2}^g, d_{i2}^g)$, $(m_{i3}, w_{i2}^g, d_{i2}^g), (m_{i3}, w_{i1}^g, d_{i1}^g)$. It can be easily verified that they do not block M' because the orders of the three dopplegangers in the simple lists of w_{i1}^g and d_{i1}^g (and also w_{i2}^g and d_{i2}^g) are reversed. □

Lemma 7. *(Necessity) Suppose that there is a perfect matching M in the original three-dimensional matching instance Υ. There also exists a stable matching M' in the derived stable family problem instance Υ'.*

Proof. We build a stable matching M' in Υ' as follows. Let the dummy players $\{m_j^\#, w_j^\#, d_j^\#\}, 1 \leq j \leq 3$, be matched to one another. Given any doppleganger m_{ij}, let his guard players $\{m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}\}, \{m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}\}$ be matched to one another as well. Furthermore, suppose that $(m_i, w_{ix}, d_{ix}) \in M$. Let the doppleganger who lists w_{ix} and d_{ix} above his guard players be matched to w_{ix} and d_{ix}, while the other two dopplegangers be matched to the garbage collectors. By this construction, it can be seen that none of the guard players and dummy players will be part of a blocking triple. This, combined with Lemma 6, completes the proof. □

Suppose that in the given three-dimensional matching instance Υ, $|\mathcal{M}| = |\mathcal{W}| = |\mathcal{D}| = n$. Then in the derived instance Υ', we use in all $3n$ dopplegangers, $18n$ guard players, $4n$ garbage collectors, $2n$ real women and real dogs, and 9 dummy players. Their preferences (in the form of simple lists) can be generated in $O(n^2)$ time. Therefore, this is a polynomial-time reduction. Also, given any matching, we definitely can check its stability in $O(n^3)$ time. Combining the two facts with Lemma 5 and Lemma 7, we can conclude:

Theorem 1. *It is NP-complete to decide whether strong stable matchings exist under the **PON** scheme. Therefore, the question of deciding existence of strong stable matching is also NP-complete when the full preference lists are consistent, i.e., when they are relaxed linear extensions of preference posets.*

Super Stability and Ultra Stability It can be observed that throughout the proof, all arguments involving blocking triples use those of degree 3. The only exception is the occasion that we argue that a doppleganger cannot be matched to his guard players in a stable matching. To recall, supposing that $(m_{ij}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1})$ is part of a matching, then $(m_{ij}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2})$ is a blocking triple of degree 2. (Or if the latter is part of the matching, the former is a blocking triple of degree 2). Therefore, our reduction only uses blocking triples of degree 2 or 3; both are still blocking triples with regard to super stability and ultra stability. Moreover, when we argue the strong-stability of matchings in the reduction, we never allow blocking triples of degree 0 or degree 1 to exist. Therefore, essentially, our reduction has also established the NP-completeness of super stable matchings and ultra stable matchings.

4 Threesome Roommates with Relaxed Linear Extensions of Preference Posets

In this section, we exhibit a reduction of stable family to threesome roommates, thereby establishing the NP-completeness of strong/super/ultra stable matchings in the latter problem. Instead of the **PON** scheme, we use the more general scheme in which any relaxed linear extension of preference posets is allowed. We choose to use this scheme because the involved reduction technique has a different flavor. Nonetheless, we do have another reduction for the **PON** scheme, whose idea is sketched in the appendix.

Let an instance of stable family problem be $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \Psi)$, where Ψ represents the preferences of the players in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$. We create an instance of threesome roommates $\Upsilon' = (\mathcal{R}', \Psi')$ by copying all players in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ into $\mathcal{R}'$. Regarding the preferences in Ψ', we first build up the simple lists of all players.

– Suppose $m \in \mathcal{M}, L(m) = L_{\mathcal{W}}(m) \succ L_{\mathcal{D}}(m) \succ \pi(\mathcal{M} - \{m\})$.
– Suppose $w \in \mathcal{W}, L(w) = L_{\mathcal{D}}(w) \succ L_{\mathcal{M}}(w) \succ \pi(\mathcal{W} - \{w\})$.
– Suppose $d \in \mathcal{D}, L(d) = L_{\mathcal{M}}(d) \succ L_{\mathcal{W}}(d) \succ \pi(\mathcal{D} - \{d\})$.

In words, a man lists all women and then all dogs, based respectively on their original order in his simple lists in Ψ. He then attaches other fellow men in arbitrary order to the end of his list. Women and dogs have analogous arrangements in their simple lists.

Having constructed the simple lists, we still need to build consistent relaxed linear extensions. By Lemma 2, we can construct them as follows:

– Consider $m \in \mathcal{M}$ and assume that $W = L_{\mathcal{W}}(m), D = L_{\mathcal{D}}(m), N = \pi(\mathcal{M} - \{m\})$. His relaxed linear extension is: $E_\pi(W \times W) \succ X \succ E_\pi(W \times N) \succ E_\pi(D \times D) \succ E_\pi(D \times N) \succ E_\pi(N \times N)$, where X is the original relaxed linear extension of man m's preference poset given in Ψ.
– Consider $w \in \mathcal{W}$ and assume that $D = L_{\mathcal{D}}(w), N = L_{\mathcal{M}}(w), W = \pi(\mathcal{W} - \{w\})$. Her relaxed linear extension is: $E_\pi(D \times D) \succ Y \succ E_\pi(D \times W) \succ E_\pi(N \times N) \succ E_\pi(N \times W) \succ E_\pi(W \times W)$, where Y is the original relaxed linear extension of woman w's preference poset given in Ψ.
– Consider $d \in \mathcal{D}$ and assume that $N = L_{\mathcal{M}}(d), W = L_{\mathcal{W}}(d), D = \pi(\mathcal{D} - \{d\})$. Its relaxed linear extension is: $E_\pi(N \times N) \succ Z \succ E_\pi(N \times D) \succ E_\pi(W \times W) \succ E_\pi(W \times D) \succ E_\pi(D \times D)$, where Z is the original relaxed linear extension of dog d's preference poset given in Ψ.

To prove that the reduction Υ to Υ' is valid, we will rely heavily on the following technical lemma.

Lemma 8. *In the derived instance Υ', if a stable matching M' exists, every triple in M' must contain a man, a woman, and a dog. Moreover, suppose that in a matching M'' in Υ' in which each player gets two other types of players as roommates, then a blocking triple cannot contain two (or three) players of the same type.*

Proof. For the first part, we argue case by case.

1. If $\{m, w_i, w_j\} \in M'$, there exists another man m' who can get neither a woman-woman combination nor a woman-dog combination. By construction, m' would prefer any woman-dog combination to his assigned roommates in M'. Similarly, there exists a dog d' who gets another fellow dog in M'. Such a dog would prefer a man-woman combination to its assigned roommates in M'. Finally, woman w_i and w_j would prefer dog-man combination. Therefore, both $\{m', w_i, d'\}$ and $\{m', w_j, d'\}$ block M', a contradiction.
2. If $\{m, m_i, m_j\} \in M'$, then there exists a woman w who gets a fellow woman in M' and a dog d who gets a fellow dog in M'. Thus, woman w would prefer a dog-man combination and dog d would prefer a man-woman combination. Therefore, $\{m, w, d\}, \{m_i, w, d\}, \{m_j, w, d\}$ block M', a contradiction.
3. All other cases can be argued similarly.

For the second part, suppose that matching M'' has the stated property. Given any man m, by our construction, if there is a blocking triple containing m and in which there are two players of the same type, the only possibility of a blocking triple is $\{m, w_i, w_j\}$. However, neither w_i nor w_j would prefer such a triple, because in our construction, for a woman, a dog-man combination is better than a man-woman combination. The other potential blocking triples not involving men follow analogous arguments, thus giving us the lemma. □

It is straightforward to use Lemma 8 to prove our reduction is a valid one.

Theorem 2. *Deciding whether strong/super/ultra stable matchings exist in the threesome roommates problem is NP-complete when full preference lists are consistent, i.e., when they are relaxed linear extension of preference posets.*

5 Weak Stability of Threesome Roommates with Strictly-ordered Consistent Preference Lists

In this section, we investigate the complexity of the threesome roommates problem under the **SOCL** scheme, with the proviso that full preference lists must be strict linear extensions of preference posets. We prove that under this scheme, both the the stable family problem and the threesome roommates problem are NP-complete, thereby answering the open question posed by the anonymous reviewers of Ng and Hirschberg. We could have shown the stable family problem is NP-complete and used this fact and the reduction given in the previous section to show threesome roommates is NP-complete. However, our reduction for the former problem needs to rely on a rather complicated gadget. On the other hand, using a similar idea, the latter problem has a simpler gadget, thus we present a direct reduction for the latter. For the former problem, a formal NP-completeness proof can be found in the appendix.

The basic idea is similar to the one we used in Section 3. Suppose that the given three-dimensional matching instance is $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \mathcal{T})$, where $\mathcal{T} \subseteq \mathcal{M} \times \mathcal{W} \times \mathcal{D}$, moreover, every element in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ appears 2 or 3 times in $\mathcal{T}$. We will transform it into a threesome roommates instance $\Upsilon' = (\mathcal{R}', \Psi')$. We also pre-process the instance Υ so that every man in $\mathcal{M}$ appears in exactly three triples of $\mathcal{T}$.

Every man m_i will have three dopplegangers m_{i1}, m_{i2}, m_{i3}, two women w^g_{i1}, w^g_{i2} and two dogs d^g_{i1}, d^g_{i2} as garbage collectors. Supposing that (m_i, w_{ix}, d_{ix}) is one of the triples given in $\mathcal{T}$, a doppleganger m_{ij} in his simple list ranks $w^g_{i2}, d^g_{i2}, w^g_{i1}, d^g_{i1}, w_{ix}, d_{ix}$ the highest, followed by his associated guard players. The key difference is how to design the guard players' preferences so that m_{ij} will not get any players ranking below them (or among them) in a stable matching.

We introduce the following gadget. Let $\Upsilon^\dagger_{ij}$ be a roommate instance of only 4 students, $m^{\flat 3}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}$, and $m^{\flat 6}_{ij}$, such that no stable matchings exists in $\Upsilon^\dagger_{ij}$. An example of such an instance can be found in Table 3 in the appendix.

Given such an instance $\Upsilon^\dagger_{ij}$, if $m^{\flat 3}_{ij}$ is "removed" from $\Upsilon^\dagger_{ij}$, we surely have a stable matching, $\{\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}\}$. On the other hand, if $m^{\flat 3}_{ij}$ "exists," then any matching of $\Upsilon^\dagger_{ij}$ will have at least one blocking triple. Our plan is to embed the instance $\Upsilon^\dagger_{ij}$ into the derived instance Υ' so that if $m^{\flat 3}_{ij}$ is *not* removed, then a blocking triple involving three members from the set $\{m^{\flat 3}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ arises.

We now explain more precisely what we mean by removing $m^{\flat 3}_{ij}$ and embedding $\Upsilon^\dagger_{ij}$ into Υ'. First, we need two more guard players $m^{\flat 1}_{ij}$ and $m^{\flat 2}_{ij}$ to make sure that m_{ij} will get two players from the set $\{w^{g2}_{ij}, w^{g1}_{ij}, w_{ix}, d^{g2}_{ij}, d^{g1}_{ij}, d_{ix}\}$. This can be achived by the simple lists and proper choices of pivots. Recall that a pivot is an element that dominates all its incomparable elements in the constructed linear extension.

– $L(m_{ij}) = w^g_{i2} \succ d^g_{i2} \succ w^g_{i1} \succ d^g_{i1} \succ w_{ix} \succ d_{ix} \succ m^{\flat 1}_{ij} \succ m^{\flat 2}_{ij} \succ \cdots$
  (pivot: $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$)
– $L(m^{\flat 1}_{ij}) = m_{ij} \succ m^{\flat 2}_{ij} \succ m^{\flat 3}_{ij} \succ \cdots$
  (pivot: $\{m^{\flat 2}_{ij}, m^{\flat 3}_{ij}\}$)
– $L(m^{\flat 2}_{ij}) = m_{ij} \succ m^{\flat 1}_{ij} \succ m^{\flat 3}_{ij} \succ \cdots$
  (pivot: $\{m^{\flat 1}_{ij}, m^{\flat 3}_{ij}\}$)
– $L(m^{\flat 3}_{ij}) = m^{\flat 1}_{ij} \succ m^{\flat 2}_{ij} \succ X_3 \succ \cdots$, where X_3 is the original simple list of $m^{\flat 3}_{ij}$ in the instance $\Upsilon^{\dagger}_{ij}$. The linear extension of poset $L(m^{\flat 3}_{ij}) \times L(m^{\flat 3}_{ij})$ is $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\} \succ E_\pi((m^{\flat 1}_{ij} \succ m^{\flat 2}_{ij}) \times X_3) \succ E_\pi((m^{\flat 1}_{ij} \succ m^{\flat 2}_{ij}) \times (L(m^{\flat 3}_{ij}) - \{X_3 \cup m^{\flat 1}_{ij} \cup m^{\flat 2}_{ij}\})) \succ E_\pi(X_3 \times X_3) \succ E_\pi(X_3 \times (L(m^{\flat 3}_{ij}) - \{X_3 \cup m^{\flat 1}_{ij} \cup m^{\flat 2}_{ij}\})) \succ E_\pi((L(m^{\flat 3}_{ij}) - \{X_3 \cup m^{\flat 1}_{ij} \cup m^{\flat 2}_{ij}\}) \times (L(m^{\flat 3}_{ij}) - \{X_3 \cup m^{\flat 1}_{ij} \cup m^{\flat 2}_{ij}\}))$. (Such an extension is allowed because of Lemma 2.)
– $L(m^{\flat 4}_{ij}) = X_4 \succ \cdots, L(m^{\flat 5}_{ij}) = X_5 \succ \cdots, L(m^{\flat 6}_{ij}) = X_6 \succ \cdots$, where X_4, X_5, X_6 are the original simple lists of $m^{\flat 4}_{ij}$, $m^{\flat 5}_{ij}$, and $m^{\flat 6}_{ij}$ in $\Upsilon^{\dagger}_{ij}$, respectively. The linear extension of $L(m^{\flat 4}_{ij}) \times L(m^{\flat 4}_{ij})$ is $E_\pi(X_4 \times X_4) \succ \cdots$; similarly, the linear extension of $L(m^{\flat 5}_{ij}) \times L(m^{\flat 5}_{ij})$ and $L(m^{\flat 6}_{ij}) \times L(m^{\flat 6}_{ij})$ are $E_\pi(X_5 \times X_5) \succ \cdots$ and $E_\pi(X_6 \times X_6) \succ \cdots$, respectively. (Such extensions are allowed because of Lemma 2.)

Ideally, in a stable matching of Υ', m_{ij} will be matched to two players ranking higher than $m^{\flat 1}_{ij}$, and $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 3}_{ij}\}$ and $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ will be part of the matching. Then, $m^{\flat 3}_{ij}$ in this sense is "removed" from $\Upsilon^{\dagger}_{ij}$; because $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$ are his favorite roommates, he has no incentive to leave them to go for members in $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$. On the other hand, if $m^{\flat 3}_{ij}$ cannot get $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$ in a matching, then $m^{\flat 3}_{ij}$ will be matched to members from the set $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ and/or some other students ranking below them, disrupting the stability of the embedded instance $\Upsilon^{\dagger}_{ij}$, and hence also Υ'. This intuition is captured by the following lemma.

Lemma 9. *In a matching M' in Υ', if the guard player $m^{\flat 3}_{ij}$ is not matched to $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$, a blocking triple containing three members from the set $\{m_{ij}, m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 3}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ blocks M'. Conversely, if $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 3}_{ij}\}$, $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\} \in M$, then there is no blocking triple involving any of the six guard players of m_{ij}.*

Proof. We first consider the case $m^{\flat 3}_{ij}$ gets one of $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$ in M'. Without loss of generality, suppose $\{m^{\flat 1}_{ij}, m^{\flat 3}_{ij}, m^{\phi}\} \in M'$. There are two subcases. (1) If $m^{\phi} = m_{ij}$, then $\{m_{ij}, m^{\flat 1}_{ij}, m^{\flat 2}_{ij}\}$ blocks M'; (2) if $m^{\phi} \neq m_{ij}$, then $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 3}_{ij}\}$ blocks M'.

So we have three more cases. (1) $m^{\flat 3}_{ij}$ is matched to a student from the set $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ and another student not from the set $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$. (2) $m^{\flat 3}_{ij}$ is matched two students neither of whom is in the set $\{m^{\flat 1}_{ij}, m^{\flat 2}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$. (3) $m^{\flat 3}_{ij}$ is matched to two students from the set $\{m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$.

– In the first case, by our construction, all four persons $\{m^{\flat 3}_{ij}, m^{\flat 4}_{ij}, m^{\flat 5}_{ij}, m^{\flat 6}_{ij}\}$ prefer all the combinations of one another to their assigned roommate. So, any three of them will constitute a blocking triple to M'.

– In the second case and the third case, three students from the $\{m_{ij}^{\flat 3}, m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}\}$ are matched to one another. The situation is identical to that we have a matching M'_{ij} in the instance Υ_{ij} (in which the fourth person is left unmatched). Since Υ_{ij} does not allow stable matching, a blocking triple must exist to block M'_{ij}. By our construction, such a blocking triple blocks M' as well. This completes the first part of the lemma.

The second part of the lemma follows from the fact that $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\}$ dominates all other elements in $m_{ij}^{\flat 3}$'s preference poset $L(m_{ij}^{\flat 3}) \times L(m_{ij}^{\flat 3})$, hence $m_{ij}^{\flat 3}$ will not form a blocking triple with anyone else. Finally, since $m_{ij}^{\flat 3}$ will not be part of a blocking triple, the three guard players $m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}$ find one another ranks the highest in their simple lists (hence also in their full preference lists). They will not form a blocking triple with one another, nor with other players. This completes the proof. □

Now we will explain how the guard players guarantee that a doppleganger m_{ij} will only get players ranking higher than them in a stable matching in Υ'.

Lemma 10. *In a stable matching M' of Υ', m_{ij} must have two players as roommates ranking higher than $m_{ij}^{\flat 1}$ in his simple list $L(m_{ij})$. Moreover, the two triples $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\}$ and $\{m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}\}$ must be part of the stable matching M'.*

Proof. The following case analysis shows that m_{ij} must get two roommates ranking higher than $m_{ij}^{\flat 1}$.

– If m_{ij} gets $\{m^{\phi 1}, m^{\phi 2}\}$ and (at least) one of them ranks lower than $m_{ij}^{\flat 2}$, then either $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\} ||_{L(m_{ij}) \times L(m_{ij})} \{m^{\phi 1}, m^{\phi 2}\}$, or $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\} \succ_{L(m_{ij}) \times L(m_{ij})} \{m^{\phi 1}, m^{\phi 2}\}$. For both cases, since $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\}$ is the pivot in the linear extension of $L(m_{ij}) \times L(m_{ij})$, m_{ij} will prefer the combination $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\}$. Similarly, for $m_{ij}^{\flat 1}$ and $m_{ij}^{\flat 2}$, the combination of m_{ij} and the other ranks the highest in the linear extension of their preference posets. Therefore, $\{m_{ij}, m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\}$ blocks M'.
– Suppose that m_{ij} gets only one of $m_{ij}^{\flat 1}$ and $m_{ij}^{\flat 2}$ as roommate in M'. We argue separately.
  - Suppose that $\{m_{ij}, m_{ij}^{\flat 1}, m_{ij}^{\flat 3}\} \in M'$ or $\{m_{ij}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\} \in M'$, then $\{m_{ij}, m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\}$ blocks M', a contradiction.
  - Suppose $\{m_{ij}, m_{ij}^{\flat 1}, m_{ij}^{\phi}\} \in M'$ or $\{m_{ij}, m_{ij}^{\flat 2}, m_{ij}^{\phi}\} \in M'$, where $m_{ij}^{\phi} \neq m_{ij}^{\flat 3}$, then because $\{m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\}$ and $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 3}\}$ are the pivots in $L(m_{ij}^{\flat 1}) \times L(m_{ij}^{\flat 1})$ and in $L(m_{ij}^{\flat 2}) \times L(m_{ij}^{\flat 2})$ respectively, $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\}$ blocks M', a contradiction.
– Suppose $\{m_{ij}, m_{ij}^{\flat 1}, m_{ij}^{\flat 2}\} \in M'$, then by Lemma 9, a blocking triple containing three students from $\{m_{ij}^{\flat 3}, m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}\}$ blocks M', again a contradiction.

By the above discussion, in M', m_{ij} must get both roommates ranking higher than $m_{ij}^{\flat 1}$. Finally, if $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\} \notin M'$, they form a blocking triple; similarly, if $\{m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}\} \notin M'$, they block M'. So we have the lemma. □

We summarize the preferences in Table 2. Note that this time the garbage collectors also need their own guard players. Let $p \in \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g$, then her/its three guard players $p^{\flat 1}, p^{\flat 2}, p^{\flat 3}$ have the simple lists as follows:

– $L(p^{\flat 1}) = p \succ p^{\flat 2} \succ p^{\flat 3} \succ \cdots$ (pivot: $\{p^{\flat 2}, p^{\flat 3}\}$)
– $L(p^{\flat 2}) = p \succ p^{\flat 1} \succ p^{\flat 3} \succ \cdots$ (pivot: $\{p^{\flat 1}, p^{\flat 3}\}$)
– $L(p^{\flat 3}) = p^{\flat 1} \succ p^{\flat 2} \succ \cdots$

Table 2. The preference lists of all players in the set $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$. We assume that there exist three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$. Moreover, for any real woman $w \in \mathcal{W}$ and real dog $d \in \mathcal{D}$, let B be the part of the simple list explicitly spelt out in the table (excluding the "$\cdots$" part), the linear extension of her/its preference poset is $E_\pi(B \times B) \succ \cdots$

Player	Simple Lists	Pivot
$m_{i1} \in \mathcal{M}_1$	$L(m_{i1})_= w_{i2}^g \succ d_{i2}^g \succ w_{i1}^g \succ d_{i1}^g \succ w_{ia} \succ d_{ia} \succ m_{i1}^{\flat 1} \succ m_{i1}^{\flat 2} \succ \cdots$	$\{m_{i1}^{\flat 1}, m_{i1}^{\flat 2}\}$
$m_{i2} \in \mathcal{M}_1$	$L(m_{i2})_= w_{i2}^g \succ d_{i2}^g \succ w_{i1}^g \succ d_{i1}^g \succ w_{ib} \succ d_{ib} \succ m_{i2}^{\flat 1} \succ m_{i2}^{\flat 2} \succ \cdots$	$\{m_{i2}^{\flat 1}, m_{i2}^{\flat 2}\}$
$m_{i3} \in \mathcal{M}_1$	$L(m_{i3})_= w_{i2}^g \succ d_{i2}^g \succ w_{i1}^g \succ d_{i1}^g \succ w_{ic} \succ d_{ic} \succ m_{i3}^{\flat 1} \succ m_{i3}^{\flat 2} \succ \cdots$	$\{m_{i3}^{\flat 1}, m_{i3}^{\flat 2}\}$
$w_{i1}^g \in \mathcal{W}_1^g$	$L(w_{i1}^g) = d_{i1}^g \succ m_{i3} \succ m_{i2} \succ m_{i1} \succ w_{i1}^{g,\flat 1} \succ w_{i1}^{g,\flat 2} \succ \cdots$	$\{w_{i1}^{g,\flat 1}, w_{i1}^{g,\flat 2}\}$
$w_{i2}^g \in \mathcal{W}_2^g$	$L(w_{i2}^g) = d_{i2}^g \succ m_{i3} \succ m_{i2} \succ m_{i1} \succ w_{i2}^{g,\flat 1} \succ w_{i2}^{g,\flat 2} \succ \cdots$	$\{w_{i2}^{g,\flat 1}, w_{i2}^{g,\flat 2}\}$
$d_{i1}^g \in \mathcal{D}_1^g$	$L(d_{i1}^g) = w_{i1}^g \succ m_{i1} \succ m_{i2} \succ m_{i3} \succ d_{i1}^{g,\flat 1} \succ d_{i1}^{g,\flat 2} \succ \cdots$	$\{d_{i1}^{g,\flat 1}, d_{i1}^{g,\flat 2}\}$
$d_{i2}^g \in \mathcal{D}_2^g$	$L(d_{i2}^g) = w_{i2}^g \succ m_{i1} \succ m_{i2} \succ m_{i3} \succ d_{i2}^{g,\flat 1} \succ d_{i2}^{g,\flat 2} \succ \cdots$	$\{d_{i2}^{g,\flat 1}, d_{i2}^{g,\flat 2}\}$
$w \in \mathcal{W}$	$L(w) = \pi(\{d \vert (*, w, d) \in \mathcal{T}\}) \succ \pi(\{m_{ij} \vert (m_i, w, *) \in \mathcal{T}, w \succ_{m_{ij}} m_{ij}^{\flat 1}\}) \succ\succ \cdots$	
$d \in \mathcal{D}$	$L(d) = \pi(\{m_{ij} \vert (m_i, *, d) \in \mathcal{T}, d \succ_{m_{ij}} m_{ij}^{\flat 1}\}) \succ \pi(\{w \vert (*, w, d) \in \mathcal{T}\}) \succ \cdots$	

Lemma 11. *Suppose a stable matching M' exists in the derived threesome roommates instance Υ'. Consider the garbage collectors $w_{i1}^g, w_{i2}^g, d_{i1}^g, d_{i2}^g$ created for man $m_i \in \mathcal{M}$. Woman w_{i1}^g and dog d_{i1}^g must belong to the same triple $t_1 \in M'$ and woman w_{i2}^g and dog d_{i2}^g must belong to the same triple $t_2 \in M'$. Moreover, in t_1 and t_2, the third roommate must be one of the dopplegangers m_{i1}, m_{i2}, and m_{i3}.*

Proof. We prove this by establishing the following facts.

Fact G: *The garbage collectors $w_{i1}^g, w_{i2}^g, d_{i1}^g, d_{i2}^g$ never get two men as roommates in the stable matching M'.*

Proof: We argue the case of w_{i1}^g with two men as roommates; the remaining cases follow analogous arguments. By Lemma 10, the possible roommates of w_{i1}^g can only be from the set $\{d_{i1}^g, d_{i2}^g, m_{i1}, m_{i2},$
$m_{i3}\}$. Suppose $\{w_{i1}^g, m_{ij}, m_{ij'}\} \in M'$. Then m_{ij} has a roommate ranking lower than $m_{ij}^{\flat 1}$, contradicting Lemma 10.

Fact H: *w_{i1}^g and d_{i2}^g cannot belong to a triple in M'; similarly, w_{i2}^g and d_{i1}^g cannot belong to a triple in M'. Moreover, none of the garbage collectors can get a real woman*

$w \in \mathcal{W}$ *and a real dog* $d \in \mathcal{D}$ *in* M'

Proof: For a contradiction, suppose that w^g_{i1} and d^g_{i2} belong to the same triple. We claim that $\{w^g_{i1}, w^{g,\flat 1}_{i1}, w^{g,\flat 2}_{i1}\}$ blocks M', because of the fact that $\{w^{g,\flat 1}_{i1}, w^{g,\flat 2}_{i1}\}$ is the pivot in w^g_{i1}'s preference poset. The other cases follows analogous argument.

By Fact G and Fact H, we only need to consider the remaining possibility that in the triple $t_2 = \{w^g_{i2}, d^g_{i2}, p\}$, where the third roommate $p \notin \{m_{i1}, m_{i2}, m_{i3}\}$. In this case, $\{w^g_{i2}, d^g_2, m_{i1}\}$ blocks M'. The third roommate in the triple t_1 can be argued similarly, and so we have the lemma. □

By the Lemma 10 and Lemma 11, we establish

Lemma 12. *(Sufficiency) If there exists a stable matching* M' *in the derived threesome roommates instance* Υ'*, there exists a perfect matching* M *in the original three-dimensional matching instance* Υ*.*

We need another lemma to show the necessity.

Lemma 13. *In a matching* M' *in the derived threesome roommates problem* Υ'*, suppose that the garbage collectors of* m_i *are matched to two of the dopplegangers of* m_i*, while the remaining doppleganger* m_{ij} *is matched to a real woman and a real dog with whom* m_i *shares a triple in* $\mathcal{T}$ *in the original three-dimensional matching instance* Υ*. Then there is no blocking triple involving any player in the set* $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}^g_1 \cup \mathcal{W}^g_2 \cup \mathcal{W} \cup \mathcal{D}^g_1 \cup \mathcal{D}^g_2 \cup \mathcal{D}$*.*

Proof. We assume that $\{\{m_{i1}, w^g_{i2}, d^g_{i2}\}, \{m_{i2}, w^g_{i1}, d^g_{i1}\}, \{m_{i3}, w_{ic}, d_{ic}\}\} \subset M$. Other cases follow analogous arguments.

Fact I*: There does not exist a blocking triple of the form:* $\{m_{ij}, w^g_{i1}, p^{\phi 1}\}, \{m_{ij}, w^g_{i2}, p^{\phi 2}\}$, $\{m_{ij}, d^g_{i1}, p^{\phi 3}\}$ *and* $\{m_{ij}, d^g_{i2}, p^{\phi 4}\}$*, where* $p^{\phi 1} \neq d^g_{i1}$*,* $p^{\phi 2} \neq d^g_{i2}$*,* $p^{\phi 3} \neq w^g_{i1}$*, and* $p^{\phi 4} \neq w^g_{i2}$*.*

Proof: We only discuss the first case. By construction, suppose m_{ij} prefers $\{w^g_{i1}, p^{\phi 1}\}$, then $p^{\phi 1}$ must be in the set of $\{w^g_{i2}, d^g_{i1}, d^g_{i2}\}$. In any of the cases, $p^{\phi 1}$ prefers his assigned roommates to the combination of $\{m_{ij}, w^g_{i1}\}$. This can be observed from the fact that w^g_{i1} always ranks below $p^{\phi 1,\flat 2}$ ($p^{\phi 1}$'s guard player). So, $\{m_{ij}, w^g_{i1}, p^{\phi 1}\}$ cannot be a blocking triple.

Fact J*: The following triples cannot block* M'*:* $\{m_{i2}, w^g_{i2}, d^g_{i2}\}, \{m_{i3}, w^g_{i2}, d^g_{i2}\}$, $\{m_{i3}, w^g_{i1}, d^g_{i1}\}$.

Proof: This can be observed from the fact that the orders of the three dopplegangers in the simple lists of w^g_{i1} and d^g_{i1} (and also of w^g_{i2}, d^g_{i2}) are reversed.

Note that by Fact I and Fact J, we have ruled out the possibility that a blocking triple involves the dopplegangers or the garbage collectors.

Fact K*: There does not exist a blocking triple involving woman $w \in \mathcal{W}$ or a real dog $d \in \mathcal{D}$.*

Proof: We only consider the first case. By construction, if woman w prefers some other combination, it can be only two real dogs $d^{\phi 5}$ and $d^{\phi 6}$, or a real dog $d^{\phi 7}$ and a doppleganger $m^{\phi 8}$. In the first case, $d^{\phi 6}$ will not prefer the combination of $\{w, d^{\phi 5}\}$, because of the way we construct the linear extension of its preference poset. So $\{w, d^{\phi 5}, d^{\phi 6}\}$ cannot be a blocking triple. In the second case, $m^{\phi 8}$, being a doppleganger, by Fact I and Fact J, he cannot be part of a blocking triple. So $\{m^{\phi 8}, w, d^{\phi 7}\}$ cannot be a blocking triple.

Combining Fact I, J and K, we prove the lemma. □

Lemma 14. *(Necessity) Suppose there is a perfect matching M in the original three-dimensional matching instance Υ. Then there also exists a stable matching M' in the derived threesome roommates instance Υ'.*

Proof. We build a stable matching M' in the derived threesome roommates instance Υ' based on M.

Suppose that $(m_i, w_{ix}, d_{ix}) \in M$. Let the doppleganger who lists w_{ix} and d_{ix} above his guard players to be matched to w_{ix} and d_{ix}, while the other two dopplegangers be matched to the garbage collectors. Let the three guard players of the garbage collectors be matched to one another as well. And finally, for the six guard players of a doppleganger m_{ij}, let $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}\}$ be matched to one another; also $\{m_{ij}^{\flat 4}, m_{ij}^{\flat 5}, m_{ij}^{\flat 6}\}$ should be matched to one another.

By this construction and Lemma 10, it can be seen that none of the guard players will be part of blocking triples. This, combined with Lemma 13, completes the proof. □

Letting $n = |\mathcal{M}| = |\mathcal{W}| = |\mathcal{D}|$, we use in all $3n$ dopplegangers, $4n$ garbage collectors, $30n$ guard players, $2n$ real women and real dogs. Therefore, the reduction can be done in polynomial time. Checking whether a matching is stable also can be done in $O(n^3)$ time. This, combined with Lemma 12 and Lemma 14, gives us the proof (the threesome roommates part) of Theorem 3.

Theorem 3. *It is NP-complete to decide whether weak stable matchings exist under the* ***SOCL*** *scheme, for both the stable family and the threesome roommates problems. Hence, it is also NP-complete to decide whether a weak stable matching exists when consistent preferences are allowed to contain ties: i.e. the full preferences are relaxed linear extensions of preference posets.*

6 Conclusion and Related Problems

In this paper, we answer the open question of whether the stable family and the threesome roommates problems are NP-complete if all players have to provide consistent preference lists. We introduce a scheme in which players can express indifference on the precondition that their preferences have to be consistent. Under this scheme, a variety of stabilities are defined and we prove that all lead to NP-complete problems.

Since we have proved that the general cases of stable family and threesome roommates are NP-complete, a natural question to ask is whether there are special cases that allow polynomial time solutions. Actually, a variant of the stable family problem that can be solved efficiently does exist.

Consider the following scheme. Every player submits two simple lists. A man evaluates combinations first by the woman he gets, then by the dog; a woman first by the man she gets, then by the dog; a dog first by the man it gets, then by the woman. (Note the asymmetry). It is not hard to see that we can apply the Gale-Shapley algorithm twice to get a weak stable matching: letting the men propose to women and then propose to dogs. Women and dogs make the decision of acceptance or rejection based on their simple lists of men [2]. Merging the two matchings will give a stable matching in the stable family problem.

However, even a little twist can make the above scheme hard to solve. Suppose a man decides first based on the woman he gets and then the dog; a woman first based on the dog she gets and then on the man; a dog decides first based on the man it gets then on the woman. The Gale-Shapley algorithm no longer works [1].

Interestingly, the above scheme is reminiscent of another open problem allegedly originated by Knuth. Suppose that a man has only a simple list for women; a woman has only a simple list for dogs; a dog has only a simple list for men. This problem is called *circular stable matching*. Its complexity is still unknown. Some interesting observations on this problem can be found in [1, 3].

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A Proofs of Lemmas in Section 2

Lemma 1 *Given any poset Q and any element $q \in Q$. there exists a linear extension l of Q such that if $q||_Q q'$, then $q \succ_l q'$.*

Proof. We construct a graph whose nodes represent elements of Q and directed edges (q_i, q_j) exist if $q_j \succ_Q q_i$. We now add directed edges from all q's incomparable elements to q. We claim the graph is still acyclic. Suppose not. Then a directed cycle including q must have been created in the process. But this implies that originally, there is a path from q to one of its incomparable elements, which is impossible.

Since the new graph is still acyclic, by the well-known fact that an acyclic graph allows a linear extension, we prove the lemma. □

Lemma 2 *Let l be a strictly-ordered list. Suppose l is decomposed into nonempty contiguous sublists $(l_1, l_2, \cdots, l_k)$ such that (1) $\bigcup_{i=1}^{k} l_i = l$, (2) if $e \succ_{l_i} f$, then $e \succ_l f$, and (3) if $e \in l_i, f \in l_j, i < j$, then $e \succ_l f$. Then there exists a linear extension of $l \times l$ such that all combinations drawn from $\{l_i, l_j\}$ precede all pairs drawn from $\{l_{i'}, l_{j'}\}$, provided that $i \leq j, i' \leq j'$ and one of the following conditions holds (1) $i < i'$, (2) $i = i', j < j'$.*

Proof. Given any two (not necessarily different) sublists l_i and l_j, we can build a directed graph $G_{ij} = (V_{ij}, E_{ij})$ in which every vertex $v \in V_{ij}$ corresponds to a combination drawn from lists l_i and l_j. Directed edges in E_{ij} represent the precedence order in the poset $l_i \times l_j$, which is a sub-poset of $l \times l$. Since G_{ij} is acylic, we have a linear extension $l_{G_{ij}}$ of the elements in G_{ij}

We now use the extensions $l_{G_{ij}}$ to construct the full extension of $l \times l$. We string out all graphs G_{ij} horizontally such that the $(k^2 + k)/2$ graphs are ordered in the same way as defined in the lemma. To be precise, we list the graphs from left to right as $G_{11}, G_{12}, \cdots, G_{1k}, G_{22}, G_{23}, \cdots, G_{2k}, \cdots, G_{(k-1)(k-1)}, G_{(k-1)k}, G_{kk}$ and we can view these graphs G_{ij} as if they were some "big" vertices in another graph $\mathcal{G}$.

If in the poset $l \times l$, there exists a combination drawn from lists $\{l_i, l_j\}$ preceding another drawn from $\{l_{i'}, l_{j'}\}$, we add a directed edge into the graph $\mathcal{G}$ from vertex G_{ij} to vertex $G_{i'j'}$.

It is not hard to see that all the newly-added edges go from "right to left" across the $(k^2 + k)/2$ big vertices. This implies that the graph $\mathcal{G}$ composed of the big vertices G_{ij} is acyclic, allowing a linear extension, which can be simply the way we list the big vertices. Replacing each big vertex with the linear extension $l_{G_{ij}}$ gives the lemma. □

B Threesome Roommates under the PON scheme

We briefly explain the reduction idea under the **PON** scheme. Again we give a reduction from a stable family instance $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \Psi)$ to a threesome roommates instance Υ', assuming that the preferences given in Ψ are based on the **PON** scheme.

Supposing that $|\mathcal{M}| = n$, we create $3n$ dummy players[1] $m_1^{\#}, m_2^{\#}, \cdots, m_{3n}^{\#}$ such that every three of them must be matched to one another in a stable matching. To be precise, given $0 \leq i \leq n-1$:

- $L(m_{3i}^{\#}) = m_{3i+1}^{\#} \succ m_{3i+2}^{\#} \succ \cdots$
- $L(m_{3i+1}^{\#}) = m_{3i+2}^{\#} \succ m_{3i}^{\#} \succ \cdots$
- $L(m_{3i+2}^{\#}) = m_{3i}^{\#} \succ m_{3i+1}^{\#} \succ \cdots$

For the players in $\mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$, they need to use these dummy players to pad their preferences:

- Consider $m \in \mathcal{M}$ and assume that $W = L_{\mathcal{W}}(m), D = L_{\mathcal{D}}(m), N = \pi(\mathcal{M} - \{m\})$. His simple list in Ψ' is $L(m) = W \succ D \succ \pi(\{m_i^{\#} | 1 \leq i \leq 3n\}) \succ N$.
- Consider $w \in \mathcal{W}$ and assume that $D = L_{\mathcal{D}}(w), N = L_{\mathcal{M}}(w), W = \pi(\mathcal{W} - \{w\})$. Her simple list in Ψ' is $L(w) = D \succ N \succ \pi(\{m_i^{\#} | 1 \leq i \leq 3n\}) \succ W$.
- Consider $d \in \mathcal{D}$ and assume that $N = L_{\mathcal{M}}(d), W = L_{\mathcal{W}}(d), D = \pi(\mathcal{D} - \{d\})$. Its simple list in Ψ' is $L(d) = N \succ W \succ \pi(\{m_i^{\#} | 1 \leq i \leq 3n\}) \succ D$.

The correctness arguments of the reduction are similar to those used in Lemma 8 and Theorem 2.

[1] The number $3n$ is actually unnecessarily big, but we use it for ease of presentation

C An Example of a Threesome Roommates Instance of Size 4 without Stable Matchings

Table 3. An example of a threesome roommates instance of size 4 such that no stable matching exists. Note in this case, there is only one possible linear extension of the preference poset.

Player	Simple Lists
$m_{ij}^{\flat 3}$	$m_{ij}^{\flat 6} \succ m_{ij}^{\flat 5} \succ m_{ij}^{\flat 4}$
$m_{ij}^{\flat 4}$	$m_{ij}^{\flat 3} \succ m_{ij}^{\flat 6} \succ m_{ij}^{\flat 5}$
$m_{ij}^{\flat 5}$	$m_{ij}^{\flat 4} \succ m_{ij}^{\flat 3} \succ m_{ij}^{\flat 6}$
$m_{ij}^{\flat 6}$	$m_{ij}^{\flat 5} \succ m_{ij}^{\flat 4} \succ m_{ij}^{\flat 3}$

D Weak Stability of Stable Family with Strictly-ordered Consistent Preference Lists under the SOCL scheme

To prove that the existence of weak stable matchings in stable family is NP-complete under the **SOCL** scheme, we again resort to the reduction of three-dimensional matching. The setting of the given instance $\Upsilon = (\mathcal{M}, \mathcal{W}, \mathcal{D}, \mathcal{T})$ and the pre-processing step are the same as we have done in Section 3 and Section 5. Every man m_i appears three times in triples in $\mathcal{T}$; and we create three dopplegangers m_{i1}, m_{i2}, m_{i3}, and four garbage collectors $w_{i1}^g, w_{i2}^g, d_{i1}^g, d_{i2}^g$. The basic idea is still to use a set of guard players to restrict the possible family members of a doppleganger m_{ij} in a stable matching. What complicates things is that we need to tailor a more involved gadget to suit our purpose.

The gadget we need, like the one we used in Section 5, is a stable family instance $\Upsilon_{ij}^{\dagger}$ without stable matchings. Thankfully, such an instance is given to us in the paper of Boros et al. [1] and is recreated in Table 4.

It can be checked that if $m_{ij}^{\flat 3}$ is removed from $\Upsilon_{ij}^{\dagger}$, then there is a stable matching $M^{\dagger} = \{(m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}), (m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2})\}$. Our plan is to embed $\Upsilon_{ij}^{\dagger}$ into the derived stable family instance Υ' so that $m_{ij}^{\flat 3}$ has to be absent from the embedded instance $\Upsilon_{ij}^{\dagger}$ in a stable matching of Υ'.

We now introduce three more guard players $m_{ij}^{\flat 0}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0}$ so that in a stable matching M' in Υ', $m_{ij}^{\flat 0}$ can be matched to $(w_{ij}^{\flat 3}, d_{ij}^{\flat 3})$. Suppose that $(m_i, w_{ix}, d_{ix}) \in \mathcal{T}$. The preference of a doppleganger m_{ij} along with his twelve guard players are as follows.

- $L_{\mathcal{W}}(m_{ij}) = w_{ij}^{g2} \succ w_{ij}^{g1} \succ w_{ix} \succ w_{ij}^{\flat 0} \succ \cdots$
  $L_{\mathcal{D}}(m_{ij}) = d_{ij}^{g2} \succ d_{ij}^{g1} \succ d_{ix} \succ d_{ij}^{\flat 0} \succ \cdots$
  (pivot: $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$)
- $L_{\mathcal{W}}(m_{ij}^{\flat 0}) = w_{ij}^{\flat 3} \succ \cdots$
  $L_{\mathcal{D}}(m_{ij}^{\flat 0}) = d_{ij}^{\flat 3} \succ \cdots$

– $L_{\mathcal{M}}(w_{ij}^{\flat 0}) = m_{ij} \succ m_{ij}^{\flat 3} \succ \cdots$
$L_{\mathcal{D}}(w_{ij}^{\flat 0}) = d_{ij}^{\flat 0} \succ \cdots$
(pivot: $(m_{ij}^{\flat 3}, d_{ij}^{\flat 0})$)
– $L_{\mathcal{M}}(d_{ij}^{\flat 0}) = m_{ij} \succ m_{ij}^{\flat 3} \succ \cdots$
$L_{\mathcal{W}}(d_{ij}^{\flat 0}) = w_{ij}^{\flat 0} \succ \cdots$
(pivot: $(m_{ij}^{\flat 3}, w_{ij}^{\flat 0})$)
– $L_{\mathcal{W}}(m_{ij}^{\flat 3}) = w_{ij}^{\flat 0} \succ X \succ \cdots$
$L_{\mathcal{D}}(m_{ij}^{\flat 3}) = d_{ij}^{\flat 0} \succ Y \succ \cdots$
(where X and Y are the simple lists of $m_{ij}^{\flat 3}$ in $\Upsilon_{ij}^{\dagger}$.)

Moreover in the linear extension of poset $L_{\mathcal{W}}(m_{ij}^{\flat 3}) \times L_{\mathcal{D}}(m_{ij}^{\flat 3})$, we make $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0}) \succ E_{\pi}(\{w_{ij}^{\flat 0}\} \times Y) \succ E_{\pi}(\{w_{ij}^{\flat 0}\} \times (L_{\mathcal{D}}(m_{ij}^{\flat 3}) - \{d_{ij}^{\flat 0} \cup Y\}) \succ E_{\pi}(X \times \{d_{ij}^{\flat 0}\}) \succ F \succ \cdots)$, where F is the same linear extension of $X \times Y$ given in $\Upsilon_{ij}^{\dagger}$.
Such a construction is allowed because of Lemma 2.
– $L_{\mathcal{M}}(w_{ij}^{\flat 3}) = X \succ m_{ij}^{\flat 0} \succ \cdots$
$L_{\mathcal{D}}(w_{ij}^{\flat 3}) = Y \succ \cdots$
(where X and Y are the simple lists of $w_{ij}^{\flat 3}$ in $\Upsilon_{ij}^{\dagger}$.)

Moreover, let the linear extension of $L_{\mathcal{M}}(w_{ij}^{\flat 3}) \times L_{\mathcal{D}}(w_{ij}^{\flat 3})$ be $F \succ E_{\pi}(\{m_{ij}^{\flat 0}\}) \times Y \succ \cdots$, where F is the same linear extension of $X \times Y$ given in $\Upsilon_{ij}^{\dagger}$.
– $L_{\mathcal{M}}(d_{ij}^{\flat 3}) = X \succ m_{ij}^{\flat 0} \succ \cdots$
$L_{\mathcal{W}}(d_{ij}^{\flat 3}) = Y \succ \cdots$
(where X and Y are the simple lists of $d_{ij}^{\flat 3}$ in $\Upsilon_{ij}^{\dagger}$.)

Moreover, let the linear extension of $L_{\mathcal{M}}(d_{ij}^{\flat 3}) \times L_{\mathcal{D}}(d_{ij}^{\flat 3})$ be $F \succ E_{\pi}(\{m_{ij}^{\flat 0}\} \times Y) \succ \cdots$, where F is the same linear extension of $X \times Y$ given in $\Upsilon_{ij}^{\dagger}$.
– For the remaining guard players $p \in \{m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}, m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}\}$, assume that her/his/its simple lists in $\Upsilon_{ij}^{\dagger}$ are X and Y respectively. For the new simple lists of p in the derived instance Υ', we attach all other players to the end of X and Y, respectively. Moreover, in the linear extension of the preference poset, we make the extension of the (subposet) $X \times Y$ identical to the one given in $\Upsilon_{ij}^{\dagger}$, moreover, the linear extension of $X \times Y$ precede all other elements.

Lemma 15. *In a matching M' in the derived stable family problem instance Υ', if the guard player $m_{ij}^{\flat 3}$ is not matched to $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$, a triple containing three members from the set $\{m_{ij}^{\flat 1}, m_{ij}^{\flat 2}, m_{ij}^{\flat 3}, w_{ij}^{\flat 1}, w_{ij}^{\flat 2}, w_{ij}^{\flat 3}, d_{ij}^{\flat 1}, d_{ij}^{\flat 2}, d_{ij}^{\flat 3}\}$ blocks the stability of M'. Conversely, if M' contains the following triple: $\{(m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}), (m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}), (m_{ij}^{\flat 0}, w_{ij}^{\flat 3}, d_{ij}^{\flat 3}), (m_{ij}^{\flat 3}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0})\}$, moreover, m_{ij} is matched to a woman and a dog ranking higher than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively, then there is no blocking triple involving any of the twelve guard players of m_{ij}.*

Proof. Suppose that $m_{ij}^{\flat 3}$ is not matched to $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$. We first rule out the possibility that $m_{ij}^{\flat 3}$ is matched to one of them. W.l.o.g., let $(m_{ij}^{\flat 3}, w_{ij}^{\flat 0}, d^{\phi}) \in M'$, where $d^{\phi} \neq d_{ij}^{\flat 0}$. Then $(m_{ij}^{\flat 3}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ blocks M', as stated in the lemma.

So now we can assume that $m_{ij}^{\flat 3}$ is matched to two players strictly ranking below $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively. We have two possible scenarios:

- All nine guard players of $\{m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}, m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}, m_{ij}^{\flat 3}, w_{ij}^{\flat 3}, d_{ij}^{\flat 3}\}$ are matched to one another in M'. This situation is identical to a matching $M_{ij}^{\dagger}$ in $\Upsilon_{ij}^{\dagger}$. Since, $\Upsilon_{ij}^{\dagger}$ allows no stable matching, at least one blocking triple involving three out of these nine guard players emerges to block $M_{ij}^{\dagger}$, and also M'.
- If some of these guard players are not matched to one another, then the situation is identical to a matching $M_{ij}^{\dagger}$ in which at least three guard players (one from each type of $\{\mathcal{M}, \mathcal{W}, \mathcal{D}\}$) are left unmatched in the instance $\Upsilon_{ij}^{\dagger}$. By the linear extensions we have constructed, they would prefer one another and form a blocking triple to $M_{ij}^{\dagger}$, and also to M'.

For the second part of the lemma, we can observe the following facts:

- $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ dominates all other elements in the linear extension of $m_{ij}^{\flat 3}$'s preference poset. Hence $m_{ij}^{\flat 3}$ has no incentive to go for any other players.
- $(m_{ij}^{\flat 3}, d_{ij}^{\flat 0})$ is the pivot in $w_{ij}^{\flat 0}$'s preference poset. Hence, the only better combination for $w_{ij}^{\flat 0}$ is $(m_{ij}, d_{ij}^{\flat 0})$. But by the statement of the lemma, m_{ij} gets family members ranking higher than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively, hence $(m_{ij}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ cannot be a blocking triple. The same argument can be applied to $d_{ij}^{\flat 0}$. So both $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ are not part of a blocking triple.
- Consider woman $w_{ij}^{\flat 3}$. If she forms a blocking triple t with other players, there are two possibilities. (1) She gets a better man in t. Such a man cannot be $m_{ij}^{\flat 3}$, as we argued previously. It can be verified that neither $m_{ij}^{\flat 1}$ nor $m_{ij}^{\flat 2}$ prefers the combination of $w_{ij}^{\flat 3}$ with any other dog player. (2) She gets the same man $m_{ij}^{\flat 0}$ but a better dog, which is either $d_{ij}^{\flat 1}$ or $d_{ij}^{\flat 2}$. It can be checked that neither $d_{ij}^{\flat 1}$ nor $d_{ij}^{\flat 2}$ prefers the combination of $w_{ij}^{\flat 3}$ and $m_{ij}^{\flat 0}$ (because of the way we construct the linear extensions of their preference posets).
  The same argument can be applied to dog $d_{ij}^{\flat 3}$. Also, $m_{ij}^{\flat 0}$ is getting his best possible combination. So he has no incentive to leave $w_{ij}^{\flat 3}$ and $d_{ij}^{\flat 3}$ either.
- The remaining players in the set $\{m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}, m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}\}$ do not form blocking triples, as can be easily verified. And this completes the proof of the second part of the lemma. □

Lemma 16. *In a stable matching M' of Υ', m_{ij} must have two players as family members ranking higher than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ in his simple lists, respectively.*

Proof. The following case analysis shows that m_{ij} must get two family members ranking higher than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively.

Table 4. An instance (where $n = 3$) of the stable family problem under the SOCL scheme that disallows any stable matching.

Player	Simple Lists	Full Preference
$m_{ij}^{\flat 1}$	$L_{\mathcal{W}}(m_{ij}^{\flat 1}) = w_{ij}^{\flat 1} \succ w_{ij}^{\flat 2} \succ w_{ij}^{\flat 3}$ $L_{\mathcal{D}}(m_{ij}^{\flat 1}) = d_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} \succ d_{ij}^{\flat 3}$	$w_{ij}^{\flat 1} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 3} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 3} \succ$ $w_{ij}^{\flat 3} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 3}$
$m_{ij}^{\flat 2}$	$L_{\mathcal{W}}(m_{ij}^{\flat 1}) = w_{ij}^{\flat 2} \succ w_{ij}^{\flat 3} \succ w_{ij}^{\flat 1}$ $L_{\mathcal{D}}(m_{ij}^{\flat 2}) = d_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} \succ d_{ij}^{\flat 3}$	$w_{ij}^{\flat 2} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 3} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 3} \succ$ $w_{ij}^{\flat 1} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 3}$
$m_{ij}^{\flat 3}$	$L_{\mathcal{W}}(m_{ij}^{\flat 1}) = w_{ij}^{\flat 1} \succ w_{ij}^{\flat 2} \succ w_{ij}^{\flat 3}$ $L_{\mathcal{D}}(m_{ij}^{\flat 2}) = d_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} \succ d_{ij}^{\flat 3}$	$w_{ij}^{\flat 1} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 1} d_{ij}^{\flat 3} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 2} d_{ij}^{\flat 3} \succ$ $w_{ij}^{\flat 3} d_{ij}^{\flat 2} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 1} \succ w_{ij}^{\flat 3} d_{ij}^{\flat 3}$
$w_{ij}^{\flat 1}$	$L_{\mathcal{D}}(w_{ij}^{\flat 1}) = d_{ij}^{\flat 1} \succ d_{ij}^{\flat 2} \succ d_{ij}^{\flat 3}$ $L_{\mathcal{M}}(w_{ij}^{\flat 1}) = m_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} \succ m_{ij}^{\flat 3}$	$d_{ij}^{\flat 1} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 1} m_{ij}^{\flat 3} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 3} \succ$ $d_{ij}^{\flat 3} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 3} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 3} d_{ij}^{\flat 3}$
$w_{ij}^{\flat 2}$	$L_{\mathcal{D}}(w_{ij}^{\flat 2}) = d_{ij}^{\flat 2} \succ d_{ij}^{\flat 3} \succ d_{ij}^{\flat 1}$ $L_{\mathcal{M}}(w_{ij}^{\flat 1}) = m_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} \succ m_{ij}^{\flat 3}$	$d_{ij}^{\flat 2} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 3} \succ d_{ij}^{\flat 3} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 3} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 3} m_{ij}^{\flat 3} \succ$ $d_{ij}^{\flat 1} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 1} d_{ij}^{\flat 3}$
$w_{ij}^{\flat 3}$	$L_{\mathcal{D}}(w_{ij}^{\flat 3}) = d_{ij}^{\flat 1} \succ d_{ij}^{\flat 2} \succ d_{ij}^{\flat 3}$ $L_{\mathcal{M}}(w_{ij}^{\flat 3}) = m_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} \succ m_{ij}^{\flat 3}$	$d_{ij}^{\flat 1} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 1} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 1} m_{ij}^{\flat 3} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 2} m_{ij}^{\flat 3} \succ$ $d_{ij}^{\flat 3} m_{ij}^{\flat 2} \succ d_{ij}^{\flat 3} m_{ij}^{\flat 1} \succ d_{ij}^{\flat 3} d_{ij}^{\flat 3}$
$d_{ij}^{\flat 1}$	$L_{\mathcal{M}}(d_{ij}^{\flat 1}) = m_{ij}^{\flat 2} \succ m_{ij}^{\flat 3} \succ m_{ij}^{\flat 1}$ $L_{\mathcal{W}}(d_{ij}^{\flat 1}) = w_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} \succ w_{ij}^{\flat 3}$	$m_{ij}^{\flat 2} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 1}$ $m_{ij}^{\flat 3} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 3}$
$d_{ij}^{\flat 2}$	$L_{\mathcal{M}}(d_{ij}^{\flat 2}) = m_{ij}^{\flat 1} \succ m_{ij}^{\flat 2} \succ m_{ij}^{\flat 3}$ $L_{\mathcal{W}}(d_{ij}^{\flat 2}) = w_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} \succ w_{ij}^{\flat 3}$	$m_{ij}^{\flat 1} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 1}$ $m_{ij}^{\flat 2} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 3}$
$d_{ij}^{\flat 3}$	$L_{\mathcal{M}}(d_{ij}^{\flat 3}) = m_{ij}^{\flat 1} \succ m_{ij}^{\flat 2} \succ m_{ij}^{\flat 3}$ $L_{\mathcal{W}}(d_{ij}^{\flat 3}) = w_{ij}^{\flat 2} \succ w_{ij}^{\flat 1} \succ w_{ij}^{\flat 3}$	$m_{ij}^{\flat 1} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 1} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 2} w_{ij}^{\flat 1}$ $m_{ij}^{\flat 2} w_{ij}^{\flat 3} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 2} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 1} \succ m_{ij}^{\flat 3} w_{ij}^{\flat 3}$

– If m_{ij} gets $(w^{\phi 1}, d^{\phi 2})$ and (at least) one of them ranks lower than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively. Then by the fact that $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ is the pivot, m_{ij} must prefer them, and so do they him, creating a blocking triple to M', a contradiction.
– Suppose that m_{ij} gets only one of $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ as family members. We claim that $(m_{ij}^{\flat 3}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ is a blocking triple. This follows from the fact that $(m_{ij}^{\flat 3}, d_{ij}^{\flat 0})$ and $(m_{ij}^{\flat 3}, w_{ij}^{\flat 0})$ are pivots in $w_{ij}^{\flat 0}$'s and $d_{ij}^{\flat 0}$'s preference posets respectively.
– Suppose $(m_{ij}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0}) \in M'$. Then $m_{ij}^{\flat 3}$ cannot get $(w_{ij}^{\flat 0}, d_{ij}^{\flat 0})$ in M' and we can apply Lemma 15 to show M' is unstable.

By the above discussion, in M', m_{ij} must get both family members ranking higher than $w_{ij}^{\flat 0}$ and $d_{ij}^{\flat 0}$ respectively, and this gives us the lemma. □

We now summarize the preferences of the players in the set $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$ in Table 5. As can be seen, their preferences are similar to those we used in Section 3. The major difference is that now each garbage collector also needs her/its own guard players. Consider any garbage collector $p \in \{\mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g\}$. We introduce the symbols $m^\star(p)$, $w^\star(p)$, and $d^\star(p)$ to represent her/its three associate guard players. Their purpose will be clear in the proofs below.

When we create the linear extension of the preference poset of a player in $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g$, supposing that Y and Z are those players ranking at least as high as their guard players, we let all elements in $Y \times Z$ precede all other elements (using Lemma 2) in the linear extensions.

Lemma 17. *Suppose that a stable matching M' exists in the derived stable family instance Υ'. Consider the garbage collectors $w^g_{i1}, w^g_{i2}, d^g_{i1}, d^g_{i2}$ created for man $m_i \in \mathcal{M}$. Then w^g_{i1} and d^g_{i1} belong to the same triple $t_1 \in M'$ and w^g_{i2} and d^g_{i2} belong to the same triple $t_2 \in M'$. Moreover, in t_1 and t_2, the third family member must be one of the dopplegangers m_{i1}, m_{i2}, m_{i3}.*

Proof. We argue first for the case of w^g_{i2}. Suppose $t_2 = (m^{\phi 1}, w^g_{i2}, d^{\phi 1})$ and $d^{\phi 1} \neq d^g_{i2}$. There are two subcases.

- If $d^{\phi 1} = d^\star(w^g_{i2})$ and $m^{\phi 1} \neq m^\star(w^g_{i2})$, then $(m^\star(w^g_{i2}), w^\star(w^g_{i2}), d^\star(w^g_{i2}))$ blocks M', a contradiction.
- If $d^{\phi 1} = d^\star(w^g_{i2})$ and $m^{\phi 1} = m^\star(w^g_{i2})$, then $(m_{i1}, w^g_{i2}, d^g_{i2})$ blocks M', again a contradiction.
- If $d^{\phi 1} \neq d^\star(w^g_{i2})$, then $(m^\star(w^g_{i2}), w^g_{i2}, d^\star(w^g_{i2}))$ blocks M'. (This is because the combination of the garbage collectors $(m^\star(w^g_{i2}), d^\star(w^g_{i2}))$ is the pivot of w^g_{i2}'s linear extension). So we have another contradiction.

Thus we have $d^{\phi 1} = d^g_{i2}$. If $m^{\phi 1} \not\in \{m_{i1}, m_{i2}, m_{i3}\}$, then $(m_{i1}, w^g_{i2}, d^g_{i2})$ blocks M'.

The case of t_1 being composed of w^g_{i1}, d^g_{i1} and another doppleganger m_{ij} follows analogous argument. Therefore, we have the lemma. □

By the previous two lemmas, we establish

Lemma 18. *(Sufficiency) If there exists a stable matching M' in the derived stable family problem instance Υ', there exists a perfect matching M in the original three-dimensional matching instance Υ.*

We need another lemma to show the necessity.

Lemma 19. *In a matching M' in the derived stable family problem instance Υ', suppose the garbage collectors of m_i are matched to two of the dopplegangers of m_i, while the remaining doppleganger m_{ij} is matched to a real woman and a real dog with whom m_i shares a triple in $\mathcal{T}$ in the original three-dimensional instance Υ. Then there is no blocking triple in which the dopplegangers m_{i1}, m_{i2}, m_{i3}, are involved.*

Proof. We assume that $\{(m_{i1}, w^g_{i2}, d^g_{i2}), (m_{i2}, w^g_{i1}, d^g_{i1}), (m_{i3}, w_{ic}, d_{ic})\} \subset M'$. Other cases follow analogous arguments.

We claim that there does not exist a blocking triple of the form $(m_{ij}, w^g_{i1}, d^{\phi 1})$, $(m_{ij}, w^g_{i2}, d^{\phi 2})$, $(m_{ij}, w^{\phi 3}, d^g_{i1})$, and $(m_{ij}, w^{\phi 4}, d^g_{i2})$ where $d^{\phi 1} \neq d^g_{i1}$, $d^{\phi 2} \neq d^g_{i2}$, $w^{\phi 3} \neq w^g_{i1}$, and $w^{\phi 4} \neq w^g_{i2}$. We only consider the first case. By the way we construct the linear extension of the preference poset of w^g_{i1}, she will prefer her original combination (m_{i2}, d^g_{i1}) over such a combination. Hence, $(m_{ij}, w^g_{i1}, d^{\phi 1})$, where $d^{\phi 1} \neq d^g_{i1}$, cannot block M'.

So we only need to consider the three following potential blocking triples: $(m_{i2}, w^g_{i2}, d^g_{i2}), (m_{i3}, w^g_{i2}, d^g_{i2}), (m_{i3}, w^g_{i1}, d^g_{i1})$. It can be easily verified that they do not block M because that the order of the three dopplegangers in the simple lists of w^g_{i1} and d^g_{i1} (and also w^g_{i2}, d^g_{i2}) are reversed. □

Lemma 20. *(Necessity) Suppose that there is a perfect matching M in the original three-dimensional matching instance Υ. Then there also exists a stable matching M' in the derived stable family instance Υ'.*

Proof. We build a stable matching M' in the derived stable family instance Υ' based on M.

Suppose that $(m_i, w_{ix}, d_{ix}) \in M$. Let the doppleganger who lists w_{ix} and d_{ix} above his guard players to be matched to w_{ix} and d_{ix}, and the other two dopplegangers be matched to the garbage collectors. The twelve guard players of m_{ij} are matched to one another as follows: $\{(m_{ij}^{\flat 1}, w_{ij}^{\flat 1}, d_{ij}^{\flat 1}), (m_{ij}^{\flat 2}, w_{ij}^{\flat 2}, d_{ij}^{\flat 2}), (m_{ij}^{\flat 0}, w_{ij}^{\flat 3}, d_{ij}^{\flat 3}), (m_{ij}^{\flat 3}, w_{ij}^{\flat 0}, d_{ij}^{\flat 0})\}$; for any garbage collector $p \in \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g$, we make her/its three guard players $(m^\star(p), w^\star(p), d^\star(p))$ a triple.

By Lemma 15, it can be seen that none of the guard players of dopplegangers will form blocking triples. Similarly, by Lemma 19, the dopplegangers will not be involved in blocking triples either. Also, since all garbage collectors are matched to players ranking higher than their guard players, their guard players also will not form blocking triples either. Combining the above facts, we complete the proof of the lemma. □

Letting $n = |\mathcal{M}| = |\mathcal{W}| = |\mathcal{D}|$, we use in all $3n$ dopplegangers, $4n$ garbage collectors, $48n$ guard players, $2n$ real women and real dogs. Therefore, the reduction can be done in polynomial time. Checking whether a matching is stable also can be done in $O(n^3)$ time. We conclude the stable family part of Theorem 3.

Table 5. The preference lists of all players in the set $X = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{W} \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g \cup \mathcal{D}$. We assume that there exist three triples $(m_i, w_{ia}, d_{ia}), (m_i, w_{ib}, d_{ib}), (m_i, w_{ic}, d_{ic})$ in $\mathcal{T}$. For $p \in \{\mathcal{W}_1^g \cup \mathcal{W}_2^g \cup \mathcal{D}_1^g \cup \mathcal{D}_2^g\}$, their linear extension should guarantee that all elements in $Y \times Z$ precede all other elements, where Y and Z are those players ranking at least as high as their guard players in their simple lists.

Player	Simple Lists	Pivot
$m_{i1} \in \mathcal{M}_1$	$L_{\mathcal{W}}(m_{i1}) = w_{i2}^g \succ w_{i1}^g \succ w_{ia} \succ w_{i1}^{\flat 0} \succ \cdots$ $L_{\mathcal{D}}(m_{i1}) = d_{i2}^g \succ d_{i1}^g \succ d_{ia} \succ d_{i1}^{\flat 0} \succ \cdots$	$(w_{i1}^{\flat 0}, d_{i1}^{\flat 0})$
$m_{i2} \in \mathcal{M}_2$	$L_{\mathcal{W}}(m_{i2}) = w_{i2}^g \succ w_{i1}^g \succ w_{ib} \succ w_{i2}^{\flat 0} \succ \cdots$ $L_{\mathcal{D}}(m_{i2}) = d_{i2}^g \succ d_{i1}^g \succ d_{ib} \succ d_{i1}^{\flat 0} \succ \cdots$	$(w_{i2}^{\flat 0}, d_{i2}^{\flat 0})$
$m_{i3} \in \mathcal{M}_3$	$L_{\mathcal{W}}(m_{i3}) = w_{i2}^g \succ w_{i1}^g \succ w_{ic} \succ w_{i3}^{\flat 0} \succ \cdots$ $L_{\mathcal{D}}(m_{i3}) = d_{i2}^g \succ d_{i1}^g \succ d_{ic} \succ d_{i3}^{\flat 0} \succ \cdots$	$(w_{i3}^{\flat 0}, d_{i3}^{\flat 0})$
$w_{i1}^g \in \mathcal{W}_1^g$	$L_{\mathcal{M}}(w_{i1}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ m^\star(w_{i1}^g) \succ \cdots$ $L_{\mathcal{D}}(w_{i1}^g) = d_{i1}^g \succ d^\star(w_{i1}^g) \succ \cdots$	$(m^\star(w_{i1}^g), d^\star(w_{i1}^g))$
$w_{i2}^g \in \mathcal{W}_2^g$	$L_{\mathcal{M}}(w_{i2}^g) = m_{i3} \succ m_{i2} \succ m_{i1} \succ m^\star(w_{i2}^g) \succ \cdots$ $L_{\mathcal{D}}(w_{i2}^g) = d_{i2}^g \succ d^\star(w_{i2}^g) \succ \cdots$	$(m^\star(w_{i2}^g), d^\star(w_{i2}^g))$
$d_{i1}^g \in \mathcal{D}_1^g$	$L_{\mathcal{M}}(d_{i1}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ m^\star(d_{i1}^g) \succ \cdots$ $L_{\mathcal{W}}(d_{i1}^g) = w_{i1}^g \succ w^\star(w_{i1}^g) \succ \cdots$	$(m^\star(d_{i1}^g), w^\star(d_{i1}^g))$
$d_{i2}^g \in \mathcal{D}_2^g$	$L_{\mathcal{M}}(d_{i2}^g) = m_{i1} \succ m_{i2} \succ m_{i3} \succ m^\star(d_{i2}^g) \succ \cdots$ $L_{\mathcal{W}}(w_{i2}^g) = w_{i2}^g \succ w^\star(w_{i2}^g) \succ \cdots$	$(m^\star(d_{i2}^g), w^\star(d_{i2}^g))$
$w \in \mathcal{W}$	$L_{\mathcal{M}}(w) = \cdots$ $L_{\mathcal{D}}(w) = \cdots$	
$d \in \mathcal{D}$	$L_{\mathcal{M}}(d) = \cdots$ $L_{\mathcal{W}}(d) = \cdots$	
$m^\star(w)$ $w \in \{w_{i1}^g, w_{i2}^g\}$	$L_{\mathcal{W}}(m^\star(w)) = w \succ w^\star(w) \succ \cdots$ $L_{\mathcal{D}}(m^\star(w)) = d^\star(w) \succ \cdots$	$(w^\star(w), d^\star(w))$
$w^\star(w)$ $w \in \{w_{i1}^g, w_{i2}^g\}$	$L_{\mathcal{M}}(w^\star(w)) = m^\star(w) \succ \cdots$ $L_{\mathcal{D}}(w^\star(w)) = d^\star(w) \succ \cdots$	
$d^\star(w)$ $w \in \{w_{i1}^g, w_{i2}^g\}$	$L_{\mathcal{M}}(d^\star(w)) = m^\star(w) \succ \cdots$ $L_{\mathcal{W}}(d^\star(w)) = w \succ w^\star(w) \succ \cdots$	$(m^\star(w), w^\star(w))$
$m^\star(d)$ $d \in \{d_{i1}^g, d_{i2}^g\}$	$L_{\mathcal{W}}(m^\star(d)) = w^\star(d) \succ \cdots$ $L_{\mathcal{D}}(m^\star(d)) = d \succ d^\star(d) \succ \cdots$	$(w^\star(d), d^\star(d))$
$w^\star(d)$ $d \in \{d_{i1}^g, d_{i2}^g\}$	$L_{\mathcal{M}}(w^\star(w)) = m^\star(d) \succ \cdots$ $L_{\mathcal{D}}(w^\star(w)) = d \succ d^\star(d) \succ \cdots$	$(m^\star(d), d^\star(d))$
$d^\star(d)$ $d \in \{d_{i1}^g, d_{i2}^g\}$	$L_{\mathcal{M}}(d^\star(d)) = m^\star(d) \succ \cdots$ $L_{\mathcal{W}}(d^\star(d)) = d^\star(d) \succ \cdots$	